

# A WEAK\*-TOPOLOGICAL DICHOTOMY WITH APPLICATIONS IN OPERATOR THEORY

TOMASZ KANIA, PIOTR KOSZMIDER, AND NIELS JAKOB LAUSTSEN

**ABSTRACT.** Denote by  $[0, \omega_1)$  the locally compact Hausdorff space consisting of all countable ordinals, equipped with the order topology, and let  $C_0[0, \omega_1)$  be the Banach space of scalar-valued, continuous functions which are defined on  $[0, \omega_1)$  and vanish eventually. We show that a weakly\* compact subset of the dual space of  $C_0[0, \omega_1)$  is either uniformly Eberlein compact, or it contains a homeomorphic copy of the ordinal interval  $[0, \omega_1]$ .

Using this result, we deduce that a Banach space which is a quotient of  $C_0[0, \omega_1)$  can either be embedded in a Hilbert-generated Banach space, or it is isomorphic to the direct sum of  $C_0[0, \omega_1)$  and a subspace of a Hilbert-generated Banach space. Moreover, we obtain a list of eight equivalent conditions describing the Loy–Willis ideal, which is the unique maximal ideal of the Banach algebra of bounded, linear operators on  $C_0[0, \omega_1)$ . As a consequence, we find that this ideal has a bounded left approximate identity, thus resolving a problem left open by Loy and Willis, and we give new proofs, in some cases of stronger versions, of several known results about the Banach space  $C_0[0, \omega_1)$  and the operators acting on it.

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The main motivation behind this paper is the desire to deepen our understanding of the Banach algebra  $\mathcal{B}(C_0[0, \omega_1))$  of (bounded, linear) operators on the Banach space  $C_0[0, \omega_1)$  of scalar-valued, continuous functions which are defined on the locally compact ordinal interval  $[0, \omega_1)$  and vanish eventually. Our strategy is to begin at a topological level, where we establish a new dichotomy for weakly\* compact subsets of the dual space of  $C_0[0, \omega_1)$ , and then use this dichotomy to obtain information about  $C_0[0, \omega_1)$  and the operators acting on it, notably a list of eight equivalent conditions characterizing the unique maximal ideal of  $\mathcal{B}(C_0[0, \omega_1))$ .

The Banach space  $C_0[0, \omega_1)$  is of course isometrically isomorphic to the hyperplane

$$\{f \in C[0, \omega_1] : f(\omega_1) = 0\}$$

---

2010 *Mathematics Subject Classification.* Primary: 46E15, 47L10; Secondary: 03E05, 46B50, 47L20, 54D30.

*Key words and phrases.* Banach space; continuous functions on the first uncountable ordinal interval; scattered space; uniform Eberlein compactness; weak\* topology; club set; stationary set; Pressing Down Lemma;  $\Delta$ -system Lemma; Banach algebra of bounded operators; maximal ideal; bounded left approximate identity.

of the Banach space  $C[0, \omega_1]$  of scalar-valued, continuous functions on the compact ordinal interval  $[0, \omega_1]$ , and hence  $C_0[0, \omega_1)$  and  $C[0, \omega_1]$  are isomorphic. Since our focus is on properties that are invariant under Banach-space isomorphism, we shall freely move between these two spaces in the following summary of the history of their study.

Semadeni [28] was the first to realize that  $C[0, \omega_1]$  is an interesting Banach space, showing that it is not isomorphic to its square, and thus producing the joint first example of an infinite-dimensional Banach space with this property. (The other example, due to Bessaga and Pełczyński [7], is James's quasi-reflexive Banach space.) The Banach-space structure of  $C_0[0, \omega_1)$  was subsequently explored in much more depth by Alspach and Benyamini [2], whose main conclusion is that  $C_0[0, \omega_1)$  is primary, in the sense that whenever  $C_0[0, \omega_1)$  is decomposed into a direct sum of two closed subspaces, one of these subspaces is necessarily isomorphic to  $C_0[0, \omega_1)$ .

Loy and Willis [22] initiated the study of the Banach algebra  $\mathcal{B}(C[0, \omega_1])$  from an automatic-continuity point of view, proving that each derivation from  $\mathcal{B}(C[0, \omega_1])$  into a Banach algebra is automatically continuous. Their result was subsequently generalized by Ogden [24], who established the automatic continuity of each algebra homomorphism from  $\mathcal{B}(C[0, \omega_1])$  into a Banach algebra.

Loy and Willis's starting point is the clever identification of a maximal ideal  $\mathcal{M}$  of co-dimension one in  $\mathcal{B}(C[0, \omega_1])$  (see equation (2.3) below for details of the definition), while their main technical step [22, Theorem 3.5] is the construction of a bounded right approximate identity in  $\mathcal{M}$ . The first- and third-named authors [18] showed recently that  $\mathcal{M}$  is the unique maximal ideal of  $\mathcal{B}(C[0, \omega_1])$ , and named it the *Loy–Willis ideal*. We shall here give a new proof of this result, together with several new characterizations of the Loy–Willis ideal. As a consequence, we obtain that  $\mathcal{M}$  has a bounded left approximate identity, thus complementing Loy and Willis's key result mentioned above.

The tools that we shall use come primarily from point-set topology and Banach space theory, and several of our results may be of independent interest to researchers in those areas, as well as to operator theorists. Before entering into a more detailed description of this paper, let us introduce four notions that will play important roles throughout.

- A topological space  $K$  is *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space; and  $K$  is *uniformly Eberlein compact* if it is homeomorphic to a weakly compact subset of a Hilbert space.
- A Banach space  $X$  is *weakly compactly generated* if it contains a weakly compact subset whose linear span is dense in  $X$ ; and  $X$  is *Hilbert-generated* if there exists a bounded operator from a Hilbert space onto a dense subspace of  $X$ .

These notions are closely related. Uniform Eberlein compactness clearly implies Eberlein compactness, and likewise Hilbert-generation implies weakly compact generation. A much deeper result, due to Amir and Lindenstrauss [3], states that a compact space  $K$  is Eberlein compact if and only if the Banach space  $C(K)$  is weakly compactly generated; and a similar relationship holds between the other two notions. Their relevance for our purposes stems primarily from the fact that the ordinal interval  $[0, \omega_1]$  is one of the “simplest” compact spaces which is not Eberlein compact.

We shall now outline how this paper is organized and state its main conclusions. Section 2 contains details of our notation, key elements of previous work, and some preliminary results. In section 3, we proceed to study the weakly\* compact subsets of the dual space of  $C_0[0, \omega_1)$ , proving in particular the following topological dichotomy.

**Theorem 1.1** (Topological Dichotomy). *Exactly one of the following two alternatives holds for each weakly\* compact subset  $K$  of  $C_0[0, \omega_1)^*$ :*

- (I) *either  $K$  is uniformly Eberlein compact; or*
- (II)  *$K$  contains a homeomorphic copy of  $[0, \omega_1]$  of the form*

$$\{\rho + \lambda \delta_\alpha : \alpha \in D\} \cup \{\rho\},$$

*where  $\rho \in C_0[0, \omega_1)^*$ ,  $\lambda$  is a non-zero scalar,  $\delta_\alpha$  is the point evaluation at  $\alpha$ , and  $D$  is a closed and unbounded subset of  $[0, \omega_1)$ .*

In Section 4, we turn our attention to the structure of operators acting on  $C_0[0, \omega_1)$ . In the case where  $T$  is a bounded, linear surjection from  $C_0[0, \omega_1)$  onto an arbitrary Banach space  $X$ , the adjoint  $T^*$  of  $T$  induces a weak\* homeomorphism of the unit ball of  $X^*$  onto a bounded subset of  $C_0[0, \omega_1)^*$ , and hence the above topological dichotomy leads to the following operator-theoretic dichotomy.

**Theorem 1.2** (Operator-theoretic Dichotomy). *Let  $X$  be a Banach space, and suppose that there exists a bounded, linear surjection  $T: C_0[0, \omega_1) \rightarrow X$ . Then exactly one of the following two alternatives holds:*

- (I) *either  $X$  embeds in a Hilbert-generated Banach space; or*
- (II) *the identity operator on  $C_0[0, \omega_1)$  factors through  $T$ , and  $X$  is isomorphic to the direct sum of  $C_0[0, \omega_1)$  and a subspace of a Hilbert-generated Banach space.*

As another consequence of Theorem 1.1, we obtain the following result.

**Theorem 1.3.** *For each bounded operator  $T$  on  $C_0[0, \omega_1)$ , there exist a unique scalar  $\varphi(T)$  and a closed and unbounded subset  $D$  of  $[0, \omega_1)$  such that*

$$(Tf)(\alpha) = \varphi(T)f(\alpha) \quad (f \in C_0[0, \omega_1), \alpha \in D). \quad (1.1)$$

*Moreover, the mapping  $T \mapsto \varphi(T)$  is linear and multiplicative, and thus a character on the Banach algebra  $\mathcal{B}(C_0[0, \omega_1))$ .*

We shall call  $\varphi$  the *Alspach–Benyamini character* because, after having discovered the above theorem, we learnt that its main part can be found in [2, p. 76, line –6].

Theorem 1.3 is the key step towards our main result: a list of eight equivalent ways of describing the Loy–Willis ideal  $\mathcal{M}$  of  $\mathcal{B}(C_0[0, \omega_1))$ . Before we can state it, another piece of notation is required. The set

$$L_0 = \bigcup_{\alpha < \omega_1} [0, \alpha] \times \{\alpha + 1\} \quad (1.2)$$

is a locally compact Hausdorff space with respect to the topology inherited from the product topology on  $[0, \omega_1)^2$ , and, as we shall see in Corollary 2.9 below, the Banach space  $C_0(L_0)$  is Hilbert-generated.

**Theorem 1.4.** *The following eight conditions are equivalent for each bounded operator  $T$  on  $C_0[0, \omega_1)$ :*

- (a)  *$T$  belongs to the Loy–Willis ideal  $\mathcal{M}$ ;*
- (b) *there is a closed and unbounded subset  $D$  of  $[0, \omega_1)$  such that  $T_{\alpha, \alpha} = 0$  for each  $\alpha \in D$ , where  $T_{\alpha, \alpha}$  denotes the  $\alpha^{\text{th}}$  diagonal entry of the matrix associated with  $T$ , as defined in equation (2.2) below;*
- (c)  *$T$  belongs to the kernel of the Alspach–Benyamini character  $\varphi$ ;*
- (d)  *$T$  factors through the Banach space  $C_0(L_0)$ ;*
- (e) *the range of  $T$  is contained in a Hilbert-generated subspace of  $C_0[0, \omega_1)$ ;*
- (f) *the range of  $T$  is contained in a weakly compactly generated subspace of  $C_0[0, \omega_1)$ ;*
- (g)  *$T$  does not fix a copy of  $C_0[0, \omega_1)$ ;*
- (h) *the identity operator on  $C_0[0, \omega_1)$  does not factor through  $T$ .*

**Remark 1.5.** (i) The equivalence of conditions (a) and (h) of Theorem 1.4 is the main result of a recent paper by the first- and third-named authors [18]. The proof that we shall give of Theorem 1.4 will not depend on that result, and thus provides an alternative proof of it.

(ii) The equivalence of conditions (a) and (d) of Theorem 1.4 disproves the conjecture stated immediately after [18, equation (5.4)].

Theorem 1.4 has a number of interesting consequences, as we shall now explain. The first, and arguably most important, of these relies on the following notion.

**Definition 1.6.** A net  $(e_\gamma)_{\gamma \in \Gamma}$  in a Banach algebra  $\mathcal{A}$  is a *bounded left approximate identity* if  $\sup_{\gamma \in \Gamma} \|e_\gamma\| < \infty$  and the net  $(e_\gamma a)_{\gamma \in \Gamma}$  converges to  $a$  for each  $a \in \mathcal{A}$ . A *bounded right approximate identity* is defined analogously, and a *bounded two-sided approximate identity* is a net which is simultaneously a bounded left and right approximate identity.

A well-known theorem of Dixon [10, Proposition 4.1] states that a Banach algebra which has both a bounded left and a bounded right approximate identity has a bounded two-sided approximate identity. As already mentioned, Loy and Willis constructed a bounded right approximate identity in  $\mathcal{M}$ . Although they did not state it formally, their result immediately raises the question whether  $\mathcal{M}$  contains a bounded left (and hence two-sided) approximate identity. We can now provide a positive answer to this question.

**Corollary 1.7.** *The Loy–Willis ideal  $\mathcal{M}$  contains a net  $(Q_D)_{D \in \Gamma}$  of projections, each having norm at most two, such that, for each operator  $T \in \mathcal{M}$ , there is  $D_0 \in \Gamma$  for which  $Q_D T = T$  whenever  $D \geq D_0$ . Hence  $(Q_D)_{D \in \Gamma}$  is a bounded left approximate identity in  $\mathcal{M}$ .*

When discovering this result, we were surprised that the net  $(Q_D T)_{D \in \Gamma}$  does not just converge to  $T$ , but it actually equals  $T$  eventually. We have, however, subsequently realized that the even stronger, two-sided counterpart of this phenomenon occurs in the unique maximal ideal of the  $C^*$ -algebra  $\mathcal{B}(\ell_2(\omega_1))$ , where  $\ell_2(\omega_1)$  denotes the first non-separable Hilbert space; see Example 4.6 for details.

Further consequences of Theorem 1.4 include generalizations of two classical Banach-space theoretic results, the first of which is Semadeni's seminal observation [28] that  $C_0[0, \omega_1)$  is not isomorphic to its square.

**Corollary 1.8.** *Let  $m, n \in \mathbb{N}$ , and suppose that  $C_0[0, \omega_1)^m$  is isomorphic to either a subspace or a quotient of  $C_0[0, \omega_1)^n$ . Then  $m \leq n$ .*

The other is Alspach and Benyamini's main theorem [2, Theorem 1] as it applies to  $C_0[0, \omega_1)$ : this Banach space is primary [2, Theorem 1].

**Corollary 1.9.** *For each bounded, linear projection  $P$  on  $C_0[0, \omega_1)$ , either the kernel of  $P$  is isomorphic to  $C_0[0, \omega_1)$  and the range of  $P$  embeds in  $C_0(L_0)$ , or vice versa.*

Another Banach-space-theoretic consequence of Theorem 1.4 is as follows; it can alternatively be deduced from [2, Lemma 1.2 and Proposition 2].

**Corollary 1.10.** *Let  $X$  be a closed subspace of  $C_0[0, \omega_1)$  such that  $X$  is isomorphic to  $C_0[0, \omega_1)$ . Then  $X$  contains a closed subspace which is complemented in  $C_0[0, \omega_1)$  and isomorphic to  $C_0[0, \omega_1)$ .*

Combining Theorem 1.4 with the techniques developed by Willis in [31], we obtain a very short proof of Ogden's main theorem [24, Theorem 6.18] as it applies to the ordinal  $\omega_1$ .

**Corollary 1.11** (Ogden). *Each algebra homomorphism from  $\mathcal{B}(C_0[0, \omega_1))$  into a Banach algebra is automatically continuous.*

Our final result relies on a suitable modification of work of the third-named author [21].

**Definition 1.12.** Let  $\mathcal{A}$  be an algebra. The *commutator* of a pair of elements  $a, b \in \mathcal{A}$  is given by  $[a, b] = ab - ba$ . A *trace* on  $\mathcal{A}$  is a scalar-valued, linear mapping  $\tau$  defined on  $\mathcal{A}$  such that  $\tau(ab) = \tau(ba)$  for each pair  $a, b \in \mathcal{A}$ .

**Corollary 1.13.** *Each operator belonging to the Loy–Willis ideal is the sum of at most three commutators.*

*Hence a scalar-valued, linear mapping  $\tau$  defined on  $\mathcal{B}(C_0[0, \omega_1))$  is a trace if and only if  $\tau$  is a scalar multiple of the Alspach–Benyamini character. In particular, each trace on  $\mathcal{B}(C_0[0, \omega_1))$  is automatically continuous.*

**Remark 1.14.** Building on Corollary 1.13, one can prove that the  $K_0$ -group of the Banach algebra  $\mathcal{B}(C_0[0, \omega_1))$  is isomorphic to  $\mathbb{Z}$  by arguments similar to those given in [20, Section 4], while the  $K_1$ -group of  $\mathcal{B}(C_0[0, \omega_1))$  vanishes. A full proof of these results will be published elsewhere [17].

## 2. PRELIMINARIES

**General conventions.** Our notation and terminology are fairly standard. We shall now outline the most important parts. Let  $X$  be a Banach space, always supposed to be over the scalar field  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . We write  $B_X$  for the closed unit ball of  $X$ . The dual space of  $X$  is  $X^*$ , and  $\langle \cdot, \cdot \rangle$  denotes the duality bracket between  $X$  and  $X^*$ .

By an *operator*, we understand a bounded, linear mapping between Banach spaces. We write  $\mathcal{B}(X)$  for the Banach algebra of all operators on  $X$ , and  $\mathcal{B}(X, Y)$  for the Banach space of all operators from  $X$  to some other Banach space  $Y$ . For an operator  $T \in \mathcal{B}(X, Y)$ , we denote by  $T^* \in \mathcal{B}(Y^*, X^*)$  its adjoint, while  $I_X$  is the identity operator on  $X$ .

Given Banach spaces  $W, X, Y$  and  $Z$  and operators  $S: W \rightarrow X$  and  $T: Y \rightarrow Z$ , we say that  $S$  *factors through*  $T$  if  $S = UTR$  for some operators  $R: W \rightarrow Y$  and  $U: Z \rightarrow X$ . The following elementary characterization of the operators that the identity operator factors through is well known.

**Lemma 2.1.** *Let  $X, Y$  and  $Z$  be Banach spaces, and let  $T: X \rightarrow Y$  be an operator. Then the identity operator on  $Z$  factors through  $T$  if and only if  $X$  contains a closed subspace  $W$  such that:*

- $W$  is isomorphic to  $Z$ ;
- the restriction of  $T$  to  $W$  is bounded below, in the sense that there exists a constant  $\varepsilon > 0$  such that  $\|Tw\| \geq \varepsilon\|w\|$  for each  $w \in W$ ;
- the image of  $W$  under  $T$  is complemented in  $Y$ .

For a Hausdorff space  $K$ ,  $C(K)$  denotes the vector space of scalar-valued, continuous functions on  $K$ . In the case where  $K$  is locally compact, we write  $C_0(K)$  for the subspace consisting of those functions  $f \in C(K)$  which ‘vanish at infinity’, in the sense that the set  $\{x \in K : |f(x)| \geq \varepsilon\}$  is compact for each  $\varepsilon > 0$ . Then  $C_0(K)$  is a Banach space with respect to the supremum norm. Alternatively, one may define  $C_0(K)$  as

$$C_0(K) = \{f \in C(\tilde{K}) : f(\infty) = 0\},$$

where  $\tilde{K} = K \cup \{\infty\}$  is the one-point compactification of  $K$ . We identify the dual space of  $C_0(K)$  with the Banach space of scalar-valued, regular Borel measures on  $K$ , and we shall therefore freely use measure-theoretic terminology and notation when dealing with functionals on  $C_0(K)$ . Given  $x \in K$ , we denote by  $\delta_x$  the Dirac measure at  $x$ .

Lower-case Greek letters such as  $\alpha, \beta, \gamma, \xi, \eta$  and  $\zeta$  denote ordinals. The first infinite ordinal is  $\omega$ , while the first uncountable ordinal is  $\omega_1$ . By convention, we consider 0 a limit ordinal. We use standard interval notation for intervals of ordinals, so that, given a pair of ordinals  $\alpha \leq \beta$ , we write  $[\alpha, \beta]$  and  $[\alpha, \beta)$  for the sets of ordinals  $\gamma$  such that  $\alpha \leq \gamma \leq \beta$  and  $\alpha \leq \gamma < \beta$ , respectively.

For a non-zero ordinal  $\alpha$ , we equip the ordinal interval  $[0, \alpha)$  with the order topology, which turns it into a locally compact Hausdorff space that is compact if and only if  $\alpha$  is a successor ordinal. (According to the standard construction of the ordinals, the interval  $[0, \alpha)$  is of course equal to the ordinal  $\alpha$ ; we use the notation  $[0, \alpha)$  to emphasize its structure as a topological space.) Since  $[0, \alpha)$  is scattered, a classical result of Rudin [27] states that each regular Borel measure on  $[0, \alpha)$  is purely atomic, so that the dual space of  $C_0[0, \alpha)$  is isometrically isomorphic to the Banach space

$$\ell_1(\alpha) = \left\{ g: [0, \alpha) \rightarrow \mathbb{K} : \sum_{\beta < \alpha} |g(\beta)| < \infty \right\}$$

via the mapping

$$g \mapsto \sum_{\beta < \alpha} g(\beta) \delta_\beta, \quad \ell_1(\alpha) \rightarrow C_0[0, \alpha]^*. \quad (2.1)$$

This implies in particular that each operator  $T$  on  $C_0[0, \alpha)$  can be represented by a scalar-valued  $[0, \alpha) \times [0, \alpha)$ -matrix  $(T_{\beta, \gamma})_{\beta, \gamma < \alpha}$  with absolutely summable rows. The  $\beta^{\text{th}}$  row of this matrix is simply the Rudin representation of the functional  $T^* \delta_\beta$ ; that is,  $(T_{\beta, \gamma})_{\gamma < \alpha}$  is the uniquely determined element of  $\ell_1(\alpha)$  such that

$$T^* \delta_\beta = \sum_{\gamma < \alpha} T_{\beta, \gamma} \delta_\gamma. \quad (2.2)$$

This matrix representation plays an essential role in the original definition of the Loy–Willis ideal, which is our next topic.

**The Loy–Willis ideal.** Suppose that  $\alpha = \omega_1 + 1$  in the notation of the previous paragraph, and note that  $C_0[0, \omega_1 + 1) = C[0, \omega_1]$ . Using the fact that each scalar-valued, continuous function on  $[0, \omega_1]$  is eventually constant, Loy and Willis [22, Proposition 3.1] proved that, for each operator  $T$  on  $C[0, \omega_1]$ , the  $\gamma^{\text{th}}$  column of its matrix,  $k_\gamma^T: \beta \mapsto T_{\beta, \gamma}$ , considered as a scalar-valued function on  $[0, \omega_1]$ , has the following three continuity properties:

- (i)  $k_\gamma^T$  is continuous whenever  $\gamma = 0$  or  $\gamma$  is a countable successor ordinal;
- (ii)  $k_\gamma^T$  is continuous at  $\omega_1$  for each countable ordinal  $\gamma$ ;
- (iii) the restriction of  $k_{\omega_1}^T$  to  $[0, \omega_1)$  is continuous, and  $\lim_{\beta \rightarrow \omega_1} k_{\omega_1}^T(\beta)$  exists.

Clause (iii) is the best possible because the final column of the matrix associated with the identity operator is equal to the indicator function  $\mathbf{1}_{\{\omega_1\}}$ , which is discontinuous at  $\omega_1$ . Hence, as Loy and Willis observed, the set

$$\mathcal{M} = \{T \in \mathcal{B}(C[0, \omega_1]) : k_{\omega_1}^T \text{ is continuous at } \omega_1\} \quad (2.3)$$

is a linear subspace of codimension one in  $\mathcal{B}(C[0, \omega_1])$ . Since the composition of operators on  $C[0, \omega_1]$  corresponds to matrix multiplication, in the sense that

$$(ST)_{\alpha, \gamma} = \sum_{\beta \leq \omega_1} S_{\alpha, \beta} T_{\beta, \gamma} \quad (S, T \in \mathcal{B}(C[0, \omega_1]), \alpha, \gamma \in [0, \omega_1]),$$

$\mathcal{M}$  is a left ideal, named the *Loy–Willis ideal* in [18]. Having codimension one,  $\mathcal{M}$  is automatically a maximal and two-sided ideal of  $\mathcal{B}(C[0, \omega_1])$ .

Consequently, the Banach algebra  $\mathcal{B}(C_0[0, \omega_1])$  also contains a maximal ideal of codimension one because it is isomorphic to  $\mathcal{B}(C[0, \omega_1])$ . Loy and Willis's definition (2.3) does not carry over to  $\mathcal{B}(C_0[0, \omega_1])$  because the matrix of an operator on  $C_0[0, \omega_1)$  has no final column. Instead we shall define the Loy–Willis ideal of  $\mathcal{B}(C_0[0, \omega_1])$  as follows. Choose an isomorphism  $U$  of  $C[0, \omega_1]$  onto  $C_0[0, \omega_1)$ , and declare that an operator  $T$  on  $C_0[0, \omega_1)$  belongs to the Loy–Willis ideal of  $\mathcal{B}(C_0[0, \omega_1])$  if and only if the operator  $U^{-1}TU$  on  $C[0, \omega_1]$  belongs to the original Loy–Willis ideal (2.3). Since the latter is a two-sided ideal, this definition is independent of the choice of the isomorphism  $U$ . We shall denote by  $\mathcal{M}$  the Loy–Willis ideal of  $\mathcal{B}(C_0[0, \omega_1])$  defined in this way; this should not cause any confusion with the original Loy–Willis ideal given by (2.3).

**Uniform Eberlein compactness.** The following theorem, which combines work of Benyamini, Rudin and Wage [5] and Benyamini and Starbird [6], collects several important characterizations of uniform Eberlein compactness.

**Theorem 2.2** (Benyamini–Rudin–Wage and Benyamini–Starbird). *The following four conditions are equivalent for a compact Hausdorff space  $K$ :*

- (a)  $K$  is uniformly Eberlein compact;
- (b) the Banach space  $C(K)$  is Hilbert-generated;
- (c) the unit ball of  $C(K)^*$  is uniformly Eberlein compact in the weak\* topology;
- (d) there exists a family  $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  of open  $F_\sigma$ -subsets of  $K$  such that:
  - (1) whenever  $x, y \in K$  are distinct, some  $G \in \mathcal{F}$  separates  $x$  and  $y$ , in the sense that either  $(x \in G \text{ and } y \notin G)$  or  $(y \in G \text{ and } x \notin G)$ ; and
  - (2)  $\sup_{x \in K} |\{G \in \mathcal{F}_n : x \in G\}|$  is finite for each  $n \in \mathbb{N}$ .

Another important theorem that we shall require is the following internal characterization of the Banach spaces which embed in a Hilbert-generated Banach space. It is closely related to the equivalence of conditions (b) and (c) above. We refer to [14, Theorem 6.30] for a proof.

**Theorem 2.3.** *A Banach space  $X$  embeds in a Hilbert-generated Banach space if and only if the unit ball of  $X^*$  is uniformly Eberlein compact in the weak\* topology.*

**The ideal of Hilbert-generated operators.** The first-named author and Kochanek [16] have recently introduced the notion of a *weakly compactly generated operator* as an operator whose range is contained in a weakly compactly generated subspace of its codomain, and have shown that the collection of all such operators forms a closed operator ideal in the sense of Pietsch. We shall now define the analogous operator ideal corresponding to the class of Hilbert-generated Banach spaces.

**Definition 2.4.** An operator  $T$  between Banach spaces  $X$  and  $Y$  is *Hilbert-generated* if its range  $T[X]$  is contained in a Hilbert-generated subspace of  $Y$ ; that is, there exist a Hilbert space  $H$  and an operator  $R: H \rightarrow Y$  such that  $T[X] \subseteq \overline{R[H]}$ . We write  $\mathcal{HG}(X, Y)$  for the set of Hilbert-generated operators from  $X$  to  $Y$ .

**Proposition 2.5.** (i) *The class  $\mathcal{HG}$  is a closed operator ideal.*

- (ii) *Let  $X$  be a Banach space. Then the ideal  $\mathcal{HG}(X)$  is proper if and only if  $X$  is not Hilbert-generated.*

*Proof.* (i). Every finite-rank operator is clearly Hilbert-generated.

Let  $W, X, Y$  and  $Z$  be Banach spaces. To see that  $\mathcal{HG}(X, Y)$  is closed under addition, suppose that  $T_1, T_2 \in \mathcal{HG}(X, Y)$ . For  $n = 1, 2$ , take a Hilbert space  $H_n$  and an operator  $R_n: H_n \rightarrow Y$  such that  $T_n[X] \subseteq \overline{R_n[H_n]}$ , and define

$$R: (x_1, x_2) \mapsto R_1 x_1 + R_2 x_2, \quad H_1 \oplus H_2 \rightarrow Y.$$

This is clearly a bounded operator with respect to the  $\ell_2$ -norm on  $H_1 \oplus H_2$ , and we have

$$\overline{R[H_1 \oplus H_2]} = \overline{R_1[H_1] + R_2[H_2]} \supseteq T_1[X] + T_2[X] = (T_1 + T_2)[X],$$

which proves that  $T_1 + T_2 \in \mathcal{HG}(X, Y)$ .



Next, given  $S \in \mathcal{B}(W, X)$ ,  $T \in \mathcal{H}\mathcal{G}(X, Y)$  and  $U \in \mathcal{B}(Y, Z)$ , take a Hilbert space  $H$  and an operator  $R: H \rightarrow Y$  such that  $T[X] \subseteq \overline{R[H]}$ . Then, by the continuity of  $U$ , we obtain  $UTS[W] \subseteq \overline{UR[H]}$ , so that  $UTS \in \mathcal{H}\mathcal{G}(W, Z)$ .

Finally, suppose that  $(T_n)_{n \in \mathbb{N}}$  is a norm-convergent sequence in  $\mathcal{H}\mathcal{G}(X, Y)$  with limit  $T$ , say. For each  $n \in \mathbb{N}$ , take a Hilbert space  $H_n$  and a contractive operator  $R_n: H_n \rightarrow Y$  such that  $T_n[X] \subseteq \overline{R_n[H_n]}$ . The Cauchy–Schwarz inequality ensures that

$$R: (x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \frac{1}{n} R_n x_n$$

defines a bounded operator from the Hilbert space  $H = (\bigoplus_{n \in \mathbb{N}} H_n)_{\ell_2}$  into  $Y$ . The fact that  $R_n[H_n] \subseteq R[H]$  implies that  $T_n[X] \subseteq \overline{R[H]}$  for each  $n \in \mathbb{N}$ , and therefore  $T[X] \subseteq \overline{R[H]}$ , which proves that  $T \in \mathcal{H}\mathcal{G}(X, Y)$ .

(ii). This is immediate because the range of the identity operator on  $X$  is contained in a Hilbert-generated subspace of  $X$  if and only if  $X$  itself is Hilbert-generated.  $\square$

**The Banach space  $C_0(L_0)$ .** The  $c_0$ -direct sum of a family  $(X_j)_{j \in J}$  of Banach spaces is given by

$$\left( \bigoplus_{j \in J} X_j \right)_{c_0} = \{ (x_j)_{j \in J} : x_j \in X_j \ (j \in J) \text{ and } \{j \in J : \|x_j\| \geq \varepsilon\} \text{ is finite for each } \varepsilon > 0 \}.$$

In the case where  $X_j = X$  for each  $j \in J$ , we write  $c_0(J, X)$  instead of  $(\bigoplus_{j \in J} X_j)_{c_0}$ . This notion is relevant for our purposes due to the following well-known elementary lemma (*e.g.*, see [9, p. 191, Exercise 9]).

**Lemma 2.6.** *Let  $L$  be the disjoint union of a family  $(L_j)_{j \in J}$  of locally compact Hausdorff spaces. Then  $C_0(L)$  is isometrically isomorphic to  $(\bigoplus_{j \in J} C_0(L_j))_{c_0}$ .*

Since the locally compact space  $L_0$  given by (1.2) is the disjoint union of the compact ordinal intervals  $[0, \alpha]$  for  $\alpha < \omega_1$ , this implies in particular that

$$C_0(L_0) \cong \left( \bigoplus_{\alpha < \omega_1} C[0, \alpha] \right)_{c_0}. \quad (2.4)$$

**Corollary 2.7.** *The Banach space  $C_0(L_0)$  is isomorphic to the  $c_0$ -direct sum of countably many copies of itself.*

*Proof.* It is well known that  $C[0, \alpha]$  is isomorphic to  $c_0(\mathbb{N}, C[0, \alpha])$  for each  $\alpha \in [\omega, \omega_1)$ , with the Banach–Mazur distance bounded uniformly in  $\alpha$  (*e.g.*, see [26, Theorem 2.24]). Hence, by (2.4), we have

$$c_0(\mathbb{N}, C_0(L_0)) \cong \left( \bigoplus_{\alpha < \omega_1} c_0(\mathbb{N}, C[0, \alpha]) \right)_{c_0} \cong \left( \bigoplus_{\alpha < \omega_1} C[0, \alpha] \right)_{c_0} \cong C_0(L_0),$$

as desired.  $\square$

**Lemma 2.8.** *Let  $(X_j)_{j \in J}$  be a family of Hilbert-generated Banach spaces. Then the Banach space  $(\bigoplus_{j \in J} X_j)_{c_0}$  is Hilbert-generated.*

*Proof.* For each  $j \in J$ , choose a Hilbert space  $H_j$  and a contractive operator  $T_j: H_j \rightarrow X_j$  with dense range. The formula  $(x_j)_{j \in J} \mapsto (T_j x_j)_{j \in J}$  then defines a contractive operator from the Hilbert space  $(\bigoplus_{j \in J} H_j)_{\ell_2}$  onto a dense subspace of  $(\bigoplus_{j \in J} X_j)_{c_0}$ .  $\square$

**Corollary 2.9.** *The Banach space  $C_0(L_0)$  is Hilbert-generated, and the one-point compactification of  $L_0$  is therefore uniformly Eberlein compact.*

*Proof.* The first statement follows immediately from (2.4) and Lemma 2.8 because  $C[0, \alpha]$  is separable and thus Hilbert-generated for each countable ordinal  $\alpha$ . Theorem 2.2 then implies the second part.  $\square$

By contrast, we have the following well-known result for  $C_0[0, \omega_1)$ .

**Theorem 2.10.** *The Banach space  $C_0[0, \omega_1)$  does not embed in any weakly compactly generated Banach space.*

*Proof.* Every weakly compactly generated Banach space is weakly Lindelöf (e.g., see [11, Theorem 12.35]), and this property is inherited by closed subspaces. However,  $C_0[0, \omega_1)$  is not weakly Lindelöf.  $\square$

Combining this result with Amir and Lindenstrauss's theorem that a compact space  $K$  is Eberlein compact if and only if  $C(K)$  is weakly compactly generated, we obtain the following conclusion, which can also be proved directly (e.g., see [11, Exercises 12.58–59]).

**Corollary 2.11.** *The ordinal interval  $[0, \omega_1]$  is not Eberlein compact.*

**Club subsets.** A subset  $D$  of  $[0, \omega_1)$  such that  $D$  is closed and unbounded is a *club subset*. Hence  $D$  is a club subset if and only if  $D$  is uncountable and  $D \cup \{\omega_1\}$  is closed in  $[0, \omega_1]$ . The collection

$$\mathcal{D} = \{D \subseteq [0, \omega_1) : D \text{ contains a club subset of } [0, \omega_1)\}$$

is a filter on the set  $[0, \omega_1)$ , and  $\mathcal{D}$  is countably complete, in the sense that  $\bigcap \mathcal{C}$  belongs to  $\mathcal{D}$  for each countable subset  $\mathcal{C}$  of  $\mathcal{D}$ .

The following lemma is a variant of [2, Lemma 1.1(c)–(d)], tailored to suit our applications. Its proof is fairly straightforward, so we omit the details.

**Lemma 2.12.** *Let  $D$  be a club subset of  $[0, \omega_1)$ .*

- (i) *The order isomorphism  $\psi_D: [0, \omega_1) \rightarrow D$  is a homeomorphism, and hence the composition operator  $U_D: g \mapsto g \circ \psi_D$  is an isometric isomorphism of  $C_0(D)$  onto  $C_0[0, \omega_1)$ .*
- (ii) *The mapping*

$$\pi_D: \alpha \mapsto \min(D \cap [\alpha, \omega_1)), \quad [0, \omega_1) \rightarrow D, \quad (2.5)$$

*is an increasing retraction, and hence the composition operator  $S_D: g \mapsto g \circ \pi_D$  is a linear isometry of  $C_0(D)$  into  $C_0[0, \omega_1)$ .*

- (iii) *Let  $\iota_D: D \rightarrow [0, \omega_1)$  denote the inclusion mapping. Then the composition operator  $R_D: f \mapsto f \circ \iota_D$  is a linear contraction of  $C_0[0, \omega_1)$  into  $C_0(D)$ , and  $R_D S_D = I_{C_0(D)}$ .*

(iv) The operator  $P_D = S_D R_D$  is a contractive projection on  $C_0[0, \omega_1)$  such that

$$\ker P_D = \{f \in C_0[0, \omega_1) : f(\alpha) = 0 \ (\alpha \in D)\}, \quad (2.6)$$

and the range of  $P_D$ ,

$$\mathcal{R}_D = \{f|_D \circ \pi_D : f \in C_0[0, \omega_1)\}, \quad (2.7)$$

is isometrically isomorphic to  $C_0[0, \omega_1)$ .

(v) Suppose that  $D \neq [0, \omega_1)$ . Then  $[0, \omega_1) \setminus D = \bigcup_{\alpha < \gamma} [\xi_\alpha, \eta_\alpha)$  for some ordinal  $\gamma \in [1, \omega_1]$  and some sequences  $(\xi_\alpha)_{\alpha < \gamma}$  and  $(\eta_\alpha)_{\alpha < \gamma}$ , where  $\xi_\alpha$  is either 0 or a countable successor ordinal,  $\eta_\alpha \in D$  and  $\xi_\alpha < \eta_\alpha < \xi_{\alpha+1}$  for each  $\alpha$ , and  $\ker P_D$  is isometrically isomorphic to  $(\bigoplus_{\alpha < \gamma} C_0[\xi_\alpha, \eta_\alpha))_{c_0}$ .

**Corollary 2.13.** For each club subset  $D$  of  $[0, \omega_1)$ ,  $\ker P_D$  is Hilbert-generated and isometrically isomorphic to a complemented subspace of  $C_0(L_0)$ .

Hence the operator  $I_{C_0[0, \omega_1)} - P_D$  belongs to the ideal  $\mathcal{H}\mathcal{G}(C_0[0, \omega_1))$ .

*Proof.* The result is trivial if  $D = [0, \omega_1)$  because  $\ker P_D = \{0\}$  in this case. Otherwise Lemma 2.12(v) applies, and the conclusions follow using Lemma 2.8 and (2.4).  $\square$

**Corollary 2.14.** There exists a club subset  $D$  of  $[0, \omega_1)$  such that  $\ker P_D$  is isometrically isomorphic to  $C_0(L_0)$ .

*Proof.* We can inductively define a transfinite sequence  $(\xi_\alpha)_{\alpha < \omega_1}$  of countable ordinals by  $\xi_0 = 0$  and  $\xi_\alpha = \sup_{\beta < \alpha} (\xi_\beta + \beta) + 2$  for each  $\alpha \in [0, \omega_1)$ . Then  $D = [0, \omega_1) \setminus \bigcup_{\alpha < \omega_1} [\xi_\alpha, \xi_\alpha + \alpha]$  is a proper club subset of  $[0, \omega_1)$ . In the notation of Lemma 2.12(v), we have  $\gamma = \omega_1$  and  $\eta_\alpha = \xi_\alpha + \alpha + 1$  for each  $\alpha < \omega_1$ , and hence

$$\ker P_D \cong \left( \bigoplus_{\alpha < \omega_1} C_0[\xi_\alpha, \eta_\alpha) \right)_{c_0} \cong \left( \bigoplus_{\alpha < \omega_1} C[0, \alpha] \right)_{c_0} \cong C_0(L_0)$$

by (2.4), as desired.  $\square$

**Lemma 2.15.** Let  $D$  and  $E$  be club subsets of  $[0, \omega_1)$ . Then

$$\ker P_D \cap \ker P_E = \ker P_{D \cup E} \quad \text{and} \quad \mathcal{R}_D \cap \mathcal{R}_E = \mathcal{R}_{D \cap E}.$$

*Proof.* The first identity is an immediate consequence of (2.6).

To verify the second, suppose first that  $f \in \mathcal{R}_D \cap \mathcal{R}_E$ . Given  $\alpha \in [0, \omega_1)$ , an easy transfinite induction shows that  $f(\beta) = f(\alpha)$  for each  $\beta \in [\alpha, \pi_{D \cap E}(\alpha)]$ , so that in particular we have  $f(\alpha) = f(\pi_{D \cap E}(\alpha))$ , and hence  $f \in \mathcal{R}_{D \cap E}$ .

Conversely, for each  $\alpha \in [0, \omega_1)$ , we see that  $D \cap [\pi_D(\alpha), \omega_1) = D \cap [\alpha, \omega_1)$ . Consequently  $D \cap E \cap [\pi_D(\alpha), \omega_1) = D \cap E \cap [\alpha, \omega_1)$ , so that  $\pi_{D \cap E}(\pi_D(\alpha)) = \pi_{D \cap E}(\alpha)$ , and therefore

$$(P_D P_{D \cap E} f)(\alpha) = f(\pi_{D \cap E}(\pi_D(\alpha))) = f(\pi_{D \cap E}(\alpha)) = (P_{D \cap E} f)(\alpha) \quad (f \in C_0[0, \omega_1)).$$

This proves that  $\mathcal{R}_{D \cap E} \subseteq \mathcal{R}_D$ . A similar argument shows that  $\mathcal{R}_{D \cap E} \subseteq \mathcal{R}_E$ .  $\square$

## 3. THE PROOF OF THEOREM 1.1

**Lemma 3.1.** *Let  $\lambda \in \mathbb{K} \setminus \{0\}$  and  $\rho \in C_0[0, \omega_1)^*$ . The mapping  $\sigma_{\lambda, \rho}: [0, \omega_1] \rightarrow C_0[0, \omega_1)^*$  given by  $\sigma_{\lambda, \rho}(\alpha) = \lambda\delta_\alpha + \rho$  for  $\alpha < \omega_1$  and  $\sigma_{\lambda, \rho}(\omega_1) = \rho$  is then injective and continuous with respect to the weak\* topology on its codomain.*

*Hence its range, which is equal to  $\{\lambda\delta_\alpha + \rho : \alpha < \omega_1\} \cup \{\rho\}$ , is homeomorphic to  $[0, \omega_1]$ .*

*Proof.* It is well known that the mapping  $\tau: [0, \omega_1] \rightarrow C_0[0, \omega_1)^*$  given by  $\tau(\alpha) = \delta_\alpha$  for  $\alpha \in [0, \omega_1)$  and  $\tau(\omega_1) = 0$  is a continuous injection with respect to the weak\* topology on its codomain, and hence the same is true for  $\sigma_{\lambda, \rho}$  because  $\lambda \neq 0$  and the vector-space operations are weakly\* continuous. The final clause now follows because  $[0, \omega_1]$  is compact, and the weak\* topology on  $C_0[0, \omega_1)^*$  is Hausdorff.  $\square$

**Definition 3.2.** A subset  $S$  of  $[0, \omega_1)$  is *stationary* if  $S \cap D \neq \emptyset$  for each club subset  $D$  of  $[0, \omega_1)$ .

Stationary sets have many interesting topological and combinatorial properties, as indicated in [15] and [19], for instance. We shall only require the following result, which is due to Fodor [12].

**Theorem 3.3** (Pressing Down Lemma). *Let  $S$  be a stationary subset of  $[0, \omega_1)$ , and let  $f: S \rightarrow [0, \omega_1)$  be a function which satisfies  $f(\alpha) < \alpha$  for each  $\alpha \in S$ . Then  $S$  contains a subset  $S'$  which is stationary and for which  $f|_{S'}$  is constant.*

We can now explain how the proof of Theorem 1.1 is structured: it consists of three parts, set out in the following lemma. Theorem 1.1 follows immediately from it, using Lemma 3.1.

**Lemma 3.4.** *Let  $K$  be a weakly\* compact subset  $K$  of  $C_0[0, \omega_1)^*$ .*

(i) *Exactly one of the following two alternatives holds:*

(I) *either there is a club subset  $D$  of  $[0, \omega_1)$  such that*

$$\mu([\alpha, \omega_1)) = 0 \quad (\mu \in K, \alpha \in D); \quad (3.1)$$

(II) *or the set*

$$\{\alpha \in [0, \omega_1) : \mu([\alpha, \omega_1)) \neq 0 \text{ for some } \mu \in K\} \quad (3.2)$$

*is stationary.*

(ii) *Condition (I) above is satisfied if and only if  $K$  is uniformly Eberlein compact.*

(iii) *Condition (II) above is satisfied if and only if there exist  $\rho \in K$ ,  $\lambda \in \mathbb{K} \setminus \{0\}$ , and a club subset  $D$  of  $[0, \omega_1)$  such that  $\rho + \lambda\delta_\alpha \in K$  for each  $\alpha \in D$ .*

In the proof, we shall require the following well-known, elementary observations.

**Lemma 3.5.** *Let  $\mu \in C_0[0, \omega_1)^*$ , and let  $\alpha \in [\omega, \omega_1)$  be a limit ordinal. Then, for each  $\varepsilon > 0$ , there exists an ordinal  $\alpha_0 < \alpha$  such that  $|\mu([\beta, \alpha))| < \varepsilon$  whenever  $\beta \in [\alpha_0, \alpha)$ .*

*Proof.* Straightforward!  $\square$

**Lemma 3.6.** (i) Let  $\{S_n : n \in \mathbb{N}\}$  be a countable family of subsets of  $[0, \omega_1)$  such that  $\bigcup_{n \in \mathbb{N}} S_n$  is a stationary subset of  $[0, \omega_1)$ . Then  $S_n$  is stationary for some  $n \in \mathbb{N}$ .  
(ii) Let  $S$  be a stationary subset of  $[0, \omega_1)$ , and let  $D$  be a club subset of  $[0, \omega_1)$ . Then  $S \cap D$  is stationary.

*Proof.* (i). Suppose contrapositively that  $S_n$  is not stationary for each  $n \in \mathbb{N}$ , and take a club subset  $D_n$  of  $[0, \omega_1)$  such that  $S_n \cap D_n = \emptyset$ . Then  $D = \bigcap_{n \in \mathbb{N}} D_n$  is a club subset of  $[0, \omega_1)$  such that  $(\bigcup_{n \in \mathbb{N}} S_n) \cap D = \emptyset$ , which shows that  $\bigcup_{n \in \mathbb{N}} S_n$  is not stationary.

(ii). We have  $(S \cap D) \cap E = S \cap (D \cap E) \neq \emptyset$  for each club subset  $E$  of  $[0, \omega_1)$  because  $D \cap E$  is a club subset.  $\square$

**Lemma 3.7.** Let  $K$  be a scattered locally compact space. Then the unit ball of  $C_0(K)^*$  is weakly\* sequentially compact.

In particular, the unit ball of  $C_0[0, \omega_1)^*$  is weakly\* sequentially compact.

*Proof.* The fact that  $K$  is scattered implies that the Banach space  $C_0(K)$  is Asplund. Consequently, the unit ball of  $C_0(K)^*$  in its weak\* topology is Radon–Nikodym compact, and thus sequentially compact.  $\square$

*Proof of Lemma 3.4.* Let  $S$  denote the set given by (3.2). Since weakly\* compact sets are bounded, we may suppose that  $K$  is contained in the unit ball of  $C_0[0, \omega_1)^*$ .

Part (i) is clear because (II) is the negation of (I).

(ii),  $\Rightarrow$ . Suppose that  $D$  is a club subset of  $[0, \omega_1)$  such that (3.1) holds. Replacing  $D$  with its intersection with the club subset of limit ordinals in  $[\omega, \omega_1)$ , we may additionally suppose that  $D$  consists entirely of infinite limit ordinals. In the case of real scalars, let  $\Delta$  be the collection of open intervals  $(q_1, q_2)$ , where  $q_1 < q_2$  are rational and  $0 \notin (q_1, q_2)$ . Otherwise  $\mathbb{K} = \mathbb{C}$ , in which case we define  $\Delta$  as the collection of open rectangles  $(q_1, q_2) \times (r_1, r_2)$  in the complex plane, where  $q_1 < q_2$  and  $r_1 < r_2$  are rational and  $0 = (0, 0) \notin (q_1, q_2) \times (r_1, r_2)$ . In both cases  $\Delta$  is countable, so that we can take a bijection  $\delta: \mathbb{N} \rightarrow \Delta$ .

For technical reasons, it is convenient to introduce a new limit ordinal, which is the predecessor of 0, and which we therefore suggestively denote by  $-1$ . Set  $D' = D \cup \{-1\}$ . For each  $\alpha \in D'$ , we define  $\alpha^+ = \pi_D(\alpha + 1) = \min(D \cap [\alpha + 1, \omega_1)) \in D$ , using the notation of Lemma 2.12(ii). Let  $\Gamma_\alpha$  denote the set of ordered pairs  $(\xi, \eta)$  of ordinals such that  $\alpha \leq \xi < \eta < \alpha^+$ , and take a bijection  $\gamma_\alpha: \mathbb{N} \rightarrow \Gamma_\alpha$ . Moreover, let  $\sigma: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  be a fixed bijection, chosen independently of  $\alpha$ . We can then define a bijection by

$$\tau_\alpha = (\gamma_\alpha \times \delta) \circ \sigma: \mathbb{N} \rightarrow \Gamma_\alpha \times \Delta \quad (\alpha \in D').$$

Hence, for each  $n \in \mathbb{N}$  and  $\alpha \in D'$ , we have  $\tau_\alpha(n) = (\xi, \eta, R)$  for some  $(\xi, \eta) \in \Gamma_\alpha$  and  $R \in \Delta$ , where  $R$  depends only on  $n$ , not on  $\alpha$ . Using this notation, we define

$$G_\alpha^n = \{\mu \in K : \mu([\xi + 1, \eta]) \in R\},$$

which is a relatively weakly\* open  $F_\sigma$ -subset of  $K$  because  $R$  is an open  $F_\sigma$ -subset of  $\mathbb{K}$  and the indicator function  $\mathbf{1}_{[\xi+1, \eta]}$  is continuous. Let  $\mathcal{F}_n = \{G_\alpha^n : \alpha \in D'\}$ . We shall now complete the proof of (ii),  $\Rightarrow$ , by verifying that the family  $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$  satisfies conditions (1)–(2) of Theorem 2.2(d).

(1). Suppose that  $\mu, \nu \in K$  are distinct. Since  $\mu$  and  $\nu$  are purely atomic, we have  $\mu(\{\alpha\}) \neq \nu(\{\alpha\})$  for some  $\alpha \in [0, \omega_1)$ . By interchanging  $\mu$  and  $\nu$  if necessary, we may suppose that  $\mu(\{\alpha\}) \neq 0$ , in which case there exists  $R \in \Delta$  such that  $\mu(\{\alpha\}) \in R$  and  $\nu(\{\alpha\}) \notin \overline{R}$ . We shall now split into two cases.

Suppose first that  $\alpha$  belongs to  $D$ . Then, as  $\alpha^+$  also belongs to  $D$ , (3.1) implies that

$$\mu([\alpha + 1, \alpha^+)) = \mu([\alpha, \omega_1)) - \mu(\{\alpha\}) - \mu([\alpha^+, \omega_1)) = 0 - \mu(\{\alpha\}) - 0 \in -R,$$

and similarly  $\nu([\alpha + 1, \alpha^+)) = -\nu(\{\alpha\}) \notin -\overline{R}$ . Since  $-R$  and the complement of  $-\overline{R}$  are open, and  $\alpha^+$  is a limit ordinal, Lemma 3.5 enables us to find  $\eta \in [\alpha + 1, \alpha^+)$  such that

$$\mu([\alpha + 1, \eta)) \in -R \quad \text{and} \quad \nu([\alpha + 1, \eta)) \notin -\overline{R}. \quad (3.3)$$

The pair  $(\alpha, \eta)$  then belongs to  $\Gamma_\alpha$ , and (3.3) shows that  $\mu \in G_\alpha^n$  and  $\nu \notin G_\alpha^n$  for  $n = \tau_\alpha^{-1}(\alpha, \eta, -R) \in \mathbb{N}$ , as desired.

Secondly, in the case where  $\alpha \notin D$  we can take  $\beta \in D'$  such that  $\beta < \alpha < \beta^+$ . (This is where the introduction of the new ordinal  $-1$  is useful.) If  $\alpha = \zeta + 1$  for some ordinal  $\zeta$ , then the pair  $(\zeta, \alpha)$  belongs to  $\Gamma_\beta$ , so that we can define  $n = \tau_\beta^{-1}(\zeta, \alpha, R) \in \mathbb{N}$ , and we have  $\mu \in G_\beta^n$  and  $\nu \notin G_\beta^n$  because  $\mu([\zeta + 1, \alpha]) = \mu(\{\alpha\}) \in R$  and  $\nu([\zeta + 1, \alpha]) = \nu(\{\alpha\}) \notin \overline{R}$ .

Otherwise  $\alpha$  is an infinite limit ordinal. By Lemma 3.5, we can find  $\xi \in [\beta, \alpha)$  such that  $\mu([\xi + 1, \alpha))$  and  $\nu([\xi + 1, \alpha))$  are as small as we like. In particular, since  $\mu(\{\alpha\})$  and  $\nu(\{\alpha\})$  belong to the open sets  $R$  and  $\mathbb{K} \setminus \overline{R}$ , respectively, we can choose  $\xi \in [\beta, \alpha)$  such that  $\mu([\xi + 1, \alpha]) \in R$  and  $\nu([\xi + 1, \alpha]) \notin \overline{R}$ . Hence the pair  $(\xi, \alpha)$  belongs to  $\Gamma_\beta$ , and we have  $\mu \in G_\beta^n$  and  $\nu \notin G_\beta^n$  for  $n = \tau_\beta^{-1}(\xi, \alpha, R) \in \mathbb{N}$ .

(2). Assume towards a contradiction that  $\sup_{\mu \in K} |\{G \in \mathcal{F}_n : \mu \in G\}|$  is infinite for some  $n \in \mathbb{N}$ , and let  $(n_1, n_2) = \sigma(n) \in \mathbb{N}^2$ . We shall focus on the case of complex scalars because it is slightly more complicated than the real case. Set  $R = \delta(n_2) = (q_1, q_2) \times (r_1, r_2) \in \Delta$ . Since  $(0, 0) \notin R$ , either  $0 \notin (q_1, q_2)$  or  $0 \notin (r_1, r_2)$ . Suppose that we are in the first case, and choose  $m \in \mathbb{N}$  such that  $m \cdot \min\{|q_1|, |q_2|\} > 1$ . By the assumption, we can find  $\mu \in K$  and ordinals  $\alpha_1 < \alpha_2 < \dots < \alpha_m$  in  $D'$  such that  $\mu \in \bigcap_{j=1}^m G_{\alpha_j}^n$ . Letting  $(\xi_j, \eta_j) = \gamma_{\alpha_j}(n_1) \in \Gamma_{\alpha_j}$  for each  $j \in \{1, \dots, m\}$ , we have  $\tau_{\alpha_j}(n) = (\xi_j, \eta_j, R)$ , so that  $\mu([\xi_j + 1, \eta_j]) \in R$  because  $\mu \in G_{\alpha_j}^n$ . The intervals  $[\xi_1 + 1, \eta_1], [\xi_2 + 1, \eta_2], \dots, [\xi_m + 1, \eta_m]$  are disjoint because

$$\alpha_1 \leq \xi_1 < \eta_1 < \alpha_1^+ \leq \alpha_2 \leq \xi_2 < \eta_2 < \alpha_2^+ \leq \dots \leq \alpha_m \leq \xi_m < \eta_m < \alpha_m^+,$$

and hence we conclude that

$$\begin{aligned} 1 &\geq \|\mu\| \geq \left| \mu \left( \bigcup_{j=1}^m [\xi_j + 1, \eta_j] \right) \right| = \left| \sum_{j=1}^m \mu([\xi_j + 1, \eta_j]) \right| \\ &\geq \left| \operatorname{Re} \sum_{j=1}^m \mu([\xi_j + 1, \eta_j]) \right| = \left| \sum_{j=1}^m \operatorname{Re} \mu([\xi_j + 1, \eta_j]) \right| \geq m \cdot \min\{|q_1|, |q_2|\} > 1, \end{aligned}$$

which is clearly absurd. The case where  $0 \notin (r_1, r_2)$  is very similar: we simply replace  $\min\{|q_1|, |q_2|\}$  and the real part with  $\min\{|r_1|, |r_2|\}$  and the imaginary part, respectively.

The case where  $\mathbb{K} = \mathbb{R}$  is also similar, but easier, because there is no need to pass to the real part in the above calculation.

(iii),  $\Leftarrow$ , is an easy consequence of the previous implications. Suppose contrapositively that condition (II) is not satisfied. Then, by (i), condition (I) holds, so that  $K$  is uniformly Eberlein compact by what we have just proved. Each weakly\* closed subset of  $K$  is therefore also uniformly Eberlein compact, and hence Corollary 2.11 implies that no subset of  $K$  is homeomorphic to  $[0, \omega_1]$ . The desired conclusion now follows from Lemma 3.1.

(iii),  $\Rightarrow$ . Suppose that the set  $S$  given by (3.2) is stationary. Since

$$S = \bigcup_{n \in \mathbb{N}} \left\{ \alpha \in [0, \omega_1) : |\mu([\alpha, \omega_1))| > \frac{1}{n} \text{ for some } \mu \in K \right\},$$

Lemma 3.6(i) implies that the set

$$S_0 = \left\{ \alpha \in [0, \omega_1) : |\mu([\alpha, \omega_1))| > \varepsilon_0 \text{ for some } \mu \in K \right\}$$

is stationary for some  $\varepsilon_0 > 0$ . Replacing  $S_0$  with its intersection with the club subset of limit ordinals in  $[\omega, \omega_1)$ , we may in addition suppose that  $S_0$  consists entirely of infinite limit ordinals by Lemma 3.6(ii). For each  $\alpha \in S_0$ , take  $\mu_\alpha \in K$  such that  $|\mu_\alpha([\alpha, \omega_1))| > \varepsilon_0$ . Lemma 3.5 implies that  $|\mu_\alpha|([f(\alpha), \alpha)) < \varepsilon_0/3$  for some ordinal  $f(\alpha) \in [0, \alpha)$ , where  $|\mu_\alpha|$  denotes the total variation of  $\mu_\alpha$ , that is, the positive measure on  $[0, \omega_1)$  given by

$$|\mu_\alpha|(B) = \sum_{\beta \in B} |\mu_\alpha(\{\beta\})| \quad (B \subseteq [0, \omega_1)).$$

By Theorem 3.3,  $S_0$  contains a subset  $S'$  which is stationary and for which  $f|_{S'}$  is constant, say  $f(\alpha) = \zeta_0$  for each  $\alpha \in S'$ .

Define  $\mathbb{L} = \mathbb{Q}$  for  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{L} = \{q + ri : q, r \in \mathbb{Q}\}$  for  $\mathbb{K} = \mathbb{C}$ , so that  $\mathbb{L}$  is a countable, dense subfield of  $\mathbb{K}$ . For each  $\alpha \in S'$  and  $k \in \mathbb{N}$ , choose a non-empty, finite subset  $F_{\alpha,k}$  of  $[0, \omega_1)$  and scalars  $q_{\alpha,k}^\beta \in \mathbb{L}$  for  $\beta \in F_{\alpha,k}$  such that

$$\mu_{\alpha,k} = \sum_{\beta \in F_{\alpha,k}} q_{\alpha,k}^\beta \delta_\beta \in C_0[0, \omega_1)^* \quad (3.4)$$

has norm at most one and satisfies

$$\|\mu_{\alpha,k} - \mu_\alpha\| < \min\left\{\frac{\varepsilon_0}{3}, \frac{1}{k}\right\}. \quad (3.5)$$

Suppose that  $(\alpha_k)_{k \in \mathbb{N}}$  is a sequence in  $S'$  such that  $(\mu_{\alpha_k,k})_{k \in \mathbb{N}}$  is weakly\* convergent with limit  $\nu \in C_0[0, \omega_1)^*$ , say. Then we claim that

$$\nu = w^*\text{-}\lim_k \mu_{\alpha_k,k}, \quad (3.6)$$

a conclusion which we shall require towards the end of the proof. Indeed, for each  $\varepsilon > 0$  and  $g \in C_0[0, \omega_1)$ , we can choose  $k_0 \in \mathbb{N}$  such that  $k_0 > 2\|g\|/\varepsilon$  and  $|\langle g, \mu_{\alpha_k,k} - \nu \rangle| < \varepsilon/2$  whenever  $k \geq k_0$ , and hence

$$|\langle g, \mu_{\alpha_k} - \nu \rangle| \leq |\langle g, \mu_{\alpha_k} - \mu_{\alpha_k,k} \rangle| + |\langle g, \mu_{\alpha_k,k} - \nu \rangle| < \frac{\|g\|}{k} + \frac{\varepsilon}{2} < \varepsilon \quad (k \geq k_0).$$

Fix  $k \in \mathbb{N}$ . Since  $\{F_{\alpha,k} : \alpha \in S'\}$  is an uncountable collection of finite sets, the  $\Delta$ -system Lemma (see [29], or [15, Theorem 9.18] for an exposition) yields the existence of a set  $\Delta_k$  and an uncountable subset  $A_k$  of  $S'$  such that

$$F_{\alpha,k} \cap F_{\beta,k} = \Delta_k \quad (\alpha, \beta \in A_k, \alpha \neq \beta). \quad (3.7)$$

We shall now arrange that a number of further properties hold by passing to suitably chosen uncountable subsets of  $A_k$ .

The fact that  $A_k = \bigcup_{n \in \mathbb{N}} \{\alpha \in A_k : |F_{\alpha,k}| = n\}$  implies that, for some  $n_k \in \mathbb{N}$ ,  $A_k$  contains an uncountable subset  $A'_k$  such that  $|F_{\alpha,k}| = n_k$  for each  $\alpha \in A'_k$ . Let  $\theta_{\alpha,k} : \{1, \dots, n_k\} \rightarrow F_{\alpha,k}$  be the unique order isomorphism for  $\alpha \in A'_k$ . Recall from (3.4) that  $q_{\alpha,k}^\beta \in \mathbb{L}$  for  $\beta \in F_{\alpha,k}$  are the coefficients of  $\mu_{\alpha,k}$ . Since  $\mathbb{L}$  is countable and

$$A'_k = \bigcup_{q_1, \dots, q_{n_k} \in \mathbb{L}} \{\alpha \in A'_k : q_{\alpha,k}^{\theta_{\alpha,k}(j)} = q_j \text{ for each } j \in \{1, \dots, n_k\}\},$$

we can find  $q_{1,k}, \dots, q_{n_k,k} \in \mathbb{L}$  and an uncountable subset  $A''_k$  of  $A'_k$  such that

$$q_{\alpha,k}^{\theta_{\alpha,k}(j)} = q_{j,k} \quad (j \in \{1, \dots, n_k\}, \alpha \in A''_k). \quad (3.8)$$

Our next aim is to show that  $A''_k$  contains an uncountable subset  $A'''_k$  such that

$$\Delta_k \subsetneq F_{\alpha,k} \quad (\alpha \in A'''_k). \quad (3.9)$$

This is trivially true if  $\Delta_k$  is empty. Otherwise let  $A'''_k = A''_k \cap [\max \Delta_k + 1, \omega_1)$ , which is uncountable, and assume towards a contradiction that  $\Delta_k = F_{\alpha,k}$  for some  $\alpha \in A'''_k$ . By (3.4), we have  $\mu_{\alpha,k}([\alpha, \omega_1)) = 0$ , so that

$$\|\mu_\alpha - \mu_{\alpha,k}\| \geq |(\mu_\alpha - \mu_{\alpha,k})([\alpha, \omega_1))| = |\mu_\alpha([\alpha, \omega_1))| > \varepsilon_0,$$

which contradicts (3.5). Hence (3.9) is satisfied for the above choice of  $A'''_k$ .

For each  $\beta \in [0, \omega_1)$ , the set

$$B_k^\beta = \{\alpha \in A'''_k : \min(F_{\alpha,k} \setminus \Delta_k) \leq \beta < \alpha\} = \bigcup_{\gamma \in [0, \beta] \setminus \Delta_k} \{\alpha \in A'''_k \cap [\gamma + 1, \omega_1) : \gamma \in F_{\alpha,k}\}$$

is countable because each of the sets on the right-hand side contains at most one element by (3.7). Hence  $A'''_k \cap [\beta + 1, \omega_1) \setminus B_k^\beta$  is uncountable, and thus non-empty; that is, for each  $\beta \in [0, \omega_1)$ , we can find  $\alpha \in A'''_k \cap [\beta + 1, \omega_1)$  such that  $\min(F_{\alpha,k} \setminus \Delta_k) > \beta$ .

Set  $\zeta_1 = \sup(\{\zeta_0\} \cup \bigcup_{j \in \mathbb{N}} \Delta_j) \in [0, \omega_1)$ . A straightforward induction based on the above observation yields a strictly increasing transfinite sequence  $(\alpha_\xi)_{\xi < \omega_1}$  in  $A'''_k \cap [\zeta_1 + 1, \omega_1)$  such that  $\sup(\{\zeta_1\} \cup \bigcup_{\eta < \xi} F_{\alpha_\eta,k} \setminus \Delta_k) < \min(F_{\alpha_\xi,k} \setminus \Delta_k)$  for each  $\xi \in [0, \omega_1)$ , and consequently  $A_k'''' = \{\alpha_\xi : \xi \in [0, \omega_1)\}$  is an uncountable subset of  $A'''_k \cap [\zeta_1 + 1, \omega_1)$  such that

$$\sup\left(\{\zeta_1\} \cup \bigcup_{\beta \in A_k'''' \cap [0, \alpha)} F_{\beta,k} \setminus \Delta_k\right) < \min(F_{\alpha,k} \setminus \Delta_k) \quad (\alpha \in A_k'''). \quad (3.10)$$



Set  $m_k = |\Delta_k| < n_k$ , and define  $\lambda_k = \sum_{j=m_k+1}^{n_k} q_{j,k} \in \mathbb{L}$ . Then, by (3.4), (3.8) and (3.10), we have  $\lambda_k = \mu_{\alpha,k}([\zeta_1, \omega_1])$  for each  $\alpha \in A_k''''$ , and hence

$$\begin{aligned} 1 &\geq \|\mu_{\alpha,k}\| \geq |\lambda_k| \geq |\mu_{\alpha}([\zeta_1, \omega_1])| - \|\mu_{\alpha,k} - \mu_{\alpha}\| \\ &\geq |\mu_{\alpha}([\alpha, \omega_1])| - |\mu_{\alpha}([\zeta_1, \alpha])| - \|\mu_{\alpha,k} - \mu_{\alpha}\| > \varepsilon_0 - \frac{\varepsilon_0}{3} - \frac{\varepsilon_0}{3} = \frac{\varepsilon_0}{3}, \end{aligned}$$

so that, after passing to a subsequence, we may suppose that  $(\lambda_k)_{k \in \mathbb{N}}$  is convergent with limit  $\lambda \in \mathbb{K}$ , say, where  $1 \geq |\lambda| \geq \varepsilon_0/3 > 0$ . (Note that, of all the estimates above, only (3.5) depends explicitly on  $k$ , and it clearly remains true after we pass to a subsequence.)

Suppose that  $\Delta_k = \{\beta_{1,k}, \dots, \beta_{m_k,k}\}$ , where  $\beta_{1,k} < \dots < \beta_{m_k,k}$ , and define

$$\rho_k = \sum_{j=1}^{m_k} q_{j,k} \delta_{\beta_{j,k}} \in C_0[0, \omega_1]^*. \quad (3.11)$$

Since  $\|\rho_k\| \leq \sum_{j=1}^{m_k} |q_{j,k}| = \|\mu_{\alpha,k}\| \leq 1$  for each  $\alpha \in A_k''''$ , Lemma 3.7 implies that, after replacing  $(\rho_k)_{k \in \mathbb{N}}$  with a subsequence, we may suppose that  $(\rho_k)_{k \in \mathbb{N}}$  is weakly\* convergent with limit  $\rho \in C_0[0, \omega_1]^*$ , say.

Our next aim is to show that, for each  $(\alpha_k)_{k \in \mathbb{N}}$  which belongs to the set

$$\begin{aligned} \mathfrak{D} = \left\{ (\alpha_k)_{k \in \mathbb{N}} : \alpha_k \in A_k'''' , \alpha_k < \alpha_{k+1} \text{ and } \max(F_{\alpha_k,k} \setminus \Delta_k) < \min(F_{\alpha_{k+1},k+1} \setminus \Delta_{k+1}) \right. \\ \left. \text{for each } k \in \mathbb{N}, \text{ and } \sup_{k \in \mathbb{N}} \alpha_k = \sup_{k \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} F_{\alpha_k,k} \setminus \Delta_k \right\}, \quad (3.12) \end{aligned}$$

the sequence  $(\mu_{\alpha_k})_{k \in \mathbb{N}}$  weakly\* converges to  $\rho + \lambda \delta_{\alpha}$ , where  $\alpha = \sup_{k \in \mathbb{N}} \alpha_k \in [0, \omega_1]$ . By (3.6), it suffices to show that  $(\mu_{\alpha_k,k})_{k \in \mathbb{N}}$  weakly\* converges to  $\rho + \lambda \delta_{\alpha}$ . To verify this, let  $\varepsilon > 0$  and  $g \in C_0[0, \omega_1]$  be given. We may suppose that  $\|g\| \leq 1$ . Choose  $k_1 \in \mathbb{N}$  such that  $|\lambda - \lambda_k| < \varepsilon/3$  and  $|\langle g, \rho - \rho_k \rangle| < \varepsilon/3$  whenever  $k \geq k_1$ . Since  $g$  is continuous at  $\alpha$ , which is a limit ordinal, we can find  $\beta_0 \in [0, \alpha]$  such that  $|g(\beta) - g(\alpha)| < \varepsilon/3$  for each  $\beta \in [\beta_0, \alpha]$ . By the definition of  $\mathfrak{D}$ , we can take  $k_2 \in \mathbb{N}$  such that  $F_{\alpha_k,k} \setminus \Delta_k \subseteq [\beta_0, \alpha]$  whenever  $k \geq k_2$ , and thus  $|g(\beta) - g(\alpha)| < \varepsilon/3$  for each  $\beta \in \bigcup_{k \geq k_2} F_{\alpha_k,k} \setminus \Delta_k$ . Now we have

$$|\langle g, \mu_{\alpha_k,k} - \rho - \lambda \delta_{\alpha} \rangle| \leq |\langle g, \mu_{\alpha_k,k} - \rho_k - \lambda_k \delta_{\alpha} \rangle| + |\langle g, \rho - \rho_k \rangle| + |\langle g, (\lambda - \lambda_k) \delta_{\alpha} \rangle|,$$

where the second and third term are both less than  $\varepsilon/3$  provided that  $k \geq k_1$ . To estimate the first term, we observe that  $\theta_{\alpha_k,k}(j) = \beta_{j,k}$  for each  $j \in \{1, \dots, m_k\}$ . Consequently (3.4), (3.8) and (3.11) imply that

$$\mu_{\alpha_k,k} - \rho_k = \sum_{j=m_k+1}^{n_k} q_{j,k} \delta_{\theta_{\alpha_k,k}(j)},$$

where  $\theta_{\alpha_k, k}(j) \in F_{\alpha_k, k} \setminus \Delta_k$  for each  $j \in \{m_k + 1, \dots, n_k\}$ , and therefore we have

$$\begin{aligned} |\langle g, \mu_{\alpha_k, k} - \rho_k - \lambda_k \delta_\alpha \rangle| &= \left| \sum_{j=m_k+1}^{n_k} q_{j, k} (g(\theta_{\alpha_k, k}(j)) - g(\alpha)) \right| \\ &\leq \sum_{j=m_k+1}^{n_k} |q_{j, k}| \cdot |g(\theta_{\alpha_k, k}(j)) - g(\alpha)| < \frac{\varepsilon}{3} \end{aligned}$$

provided that  $k \geq k_2$ . Hence we conclude that  $(\mu_{\alpha_k})_{k \in \mathbb{N}}$  weakly\* converges to  $\rho + \lambda \delta_\alpha$ .

This implies in particular that  $\rho + \lambda \delta_\alpha \in K$  for each  $\alpha$  belonging to the set

$$D = \left\{ \sup_{k \in \mathbb{N}} \alpha_k : (\alpha_k)_{k \in \mathbb{N}} \in \mathfrak{D} \right\}. \quad (3.13)$$

We shall now complete the proof by showing that  $D$  is a club subset of  $[0, \omega_1)$ . (Note that this will automatically ensure that  $\rho \in K$  because the unboundedness of  $D$  implies that the net  $(\delta_\alpha)_{\alpha \in D}$  converges weakly\* to 0, so that  $\rho = w^*\text{-}\lim_\alpha (\rho + \lambda \delta_\alpha) \in K$ .)

First, to see that  $D$  is unbounded, let  $\beta \in [0, \omega_1)$  be given. By (3.10), the transfinite sequence  $(\min(F_{\alpha_k, k} \setminus \Delta_k))_{\alpha \in A_k''''}$  is strictly increasing for each  $k \in \mathbb{N}$ , and thus unbounded. We can therefore inductively construct a sequence  $(\alpha_k)_{k \in \mathbb{N}}$  in  $[\beta, \omega_1)$  such that  $\alpha_k \in A_k''''$  and

$$\max(\{\alpha_k\} \cup (F_{\alpha_k, k} \setminus \Delta_k)) < \min(\{\alpha_{k+1}\} \cup (F_{\alpha_{k+1}, k+1} \setminus \Delta_{k+1})) \quad (k \in \mathbb{N}).$$

We claim that this sequence  $(\alpha_k)_{k \in \mathbb{N}}$  belongs to  $\mathfrak{D}$ . Of the conditions in (3.12), only the final one is not immediately obvious, and it follows from the intertwining relation

$$\alpha_k < \min(F_{\alpha_{k+1}, k+1} \setminus \Delta_{k+1}) \leq \max(F_{\alpha_{k+1}, k+1} \setminus \Delta_{k+1}) < \alpha_{k+2} \quad (k \in \mathbb{N}).$$

Consequently, we have  $\sup D \geq \sup_{k \in \mathbb{N}} \alpha_k \geq \beta$ , as desired.

Second, to verify that  $D$  is closed, we observe that each  $\beta \in \overline{D}$  is countable, and thus the limit of a sequence  $(\beta^j)_{j \in \mathbb{N}}$  in  $D$ . We may suppose that  $(\beta^j)_{j \in \mathbb{N}}$  is strictly increasing. For each  $j \in \mathbb{N}$ , take  $(\alpha_k^j)_{k \in \mathbb{N}} \in \mathfrak{D}$  such that  $\beta^j = \sup_{k \in \mathbb{N}} \alpha_k^j$ . The sequence  $(\min(\{\alpha_k^j\} \cup (F_{\alpha_k^j, k} \setminus \Delta_k)))_{k \in \mathbb{N}}$  is then strictly increasing with limit  $\beta^j$ , so we may inductively choose a strictly increasing sequence  $(k_j)_{j \in \mathbb{N}}$  of integers such that  $k_1 = 1$  and

$$\beta^j < \min(\{\alpha_{k_{j+1}}^{j+1}\} \cup (F_{\alpha_{k_{j+1}}^{j+1}, k_{j+1}} \setminus \Delta_{k_{j+1}})) \quad (j \in \mathbb{N}). \quad (3.14)$$

We now claim that the sequence  $(\gamma_\ell)_{\ell \in \mathbb{N}}$  given by

$$(\alpha_1^1, \alpha_2^1, \dots, \alpha_{k_2-1}^1, \alpha_{k_2}^2, \alpha_{k_2+1}^2, \dots, \alpha_{k_3-1}^2, \alpha_{k_3}^3, \dots, \alpha_{k_j-1}^j, \alpha_{k_j}^j, \alpha_{k_j+1}^j, \dots, \alpha_{k_{j+1}-1}^j, \alpha_{k_{j+1}}^{j+1}, \dots)$$

belongs to  $\mathfrak{D}$ . Indeed, for  $\ell \in \mathbb{N}$ , let  $j \in \mathbb{N}$  be the unique number such that  $k_j \leq \ell < k_{j+1}$ . We then have  $\gamma_\ell = \alpha_\ell^j \in A_\ell''''$ . If  $\ell \neq k_{j+1} - 1$ , then  $\gamma_{\ell+1} = \alpha_{\ell+1}^j$ , in which case the inequalities  $\gamma_\ell < \gamma_{\ell+1}$  and  $\max(F_{\gamma_\ell, \ell} \setminus \Delta_\ell) < \min(F_{\gamma_{\ell+1}, \ell+1} \setminus \Delta_{\ell+1})$  are both immediate from (3.12). Otherwise  $\ell = k_{j+1} - 1$ , and by (3.14), we find

$$\gamma_\ell = \alpha_\ell^j < \sup_{k \in \mathbb{N}} \alpha_k^j = \beta^j < \alpha_{k_{j+1}}^{j+1} = \gamma_{\ell+1}$$

and

$$\max(F_{\gamma_\ell, \ell} \setminus \Delta_\ell) < \beta^j < \min(F_{\alpha_{k_{j+1}}^{j+1}, k_{j+1}} \setminus \Delta_{k_{j+1}}) = \min(F_{\gamma_{\ell+1}, \ell+1} \setminus \Delta_{\ell+1}).$$

These intertwining relations imply that  $\sup_{\ell \in \mathbb{N}} \gamma_\ell = \beta = \sup \bigcup_{\ell \in \mathbb{N}} (F_{\gamma_\ell, \ell} \setminus \Delta_\ell)$ , which shows that  $(\gamma_\ell)_{\ell \in \mathbb{N}} \in \mathfrak{D}$ , and hence  $\beta \in D$ , as required.

(ii),  $\Leftarrow$ , now follows easily by contraposition, just as (iii),  $\Leftarrow$ , did. Indeed, suppose that condition (I) is not satisfied. Then, by (i), condition (II) is satisfied, so that Lemma 3.1 and the forward implication of (iii) imply that  $K$  contains a subset which is homeomorphic to  $[0, \omega_1]$ . Hence  $K$  is not (uniformly) Eberlein compact by Corollary 2.11.  $\square$

The idea that a result like Theorem 1.1 might be true was inspired by a note [30] from Richard Smith. The following corollary confirms a conjecture that he proposed therein.

**Corollary 3.8.** *Let  $K$  be a weakly\* compact subset of  $C_0[0, \omega_1]^*$  such that there exists a continuous surjection from  $K$  onto  $[0, \omega_1]$ . Then  $K$  contains a homeomorphic copy of  $[0, \omega_1]$ .*

*Proof.* By a classical result of Benyamini, Rudin and Wage [5], the continuous image of an Eberlein compact space is Eberlein compact. Since  $[0, \omega_1]$  is not Eberlein compact,  $K$  cannot be Eberlein compact, and we are therefore in case (II) of Theorem 1.1.  $\square$

**Example 3.9.** The purpose of this example is to show that the dichotomy stated in Lemma 3.4(i) is no longer true if condition (II) is replaced with the condition

(II') the set  $S = \{\alpha \in [0, \omega_1) : \mu([\alpha + 1, \omega_1)) \neq 0 \text{ for some } \mu \in K\}$  is stationary.

Indeed, let  $\Lambda$  be the set of all countable limit ordinals. Then  $K = \{\delta_\alpha - \delta_{\alpha+1} : \alpha \in \Lambda\} \cup \{0\}$  is a bounded and weakly\* closed subset of  $C_0[0, \omega_1]^*$ , and thus weakly\* compact. Moreover,  $K$  satisfies condition (I) because (3.1) holds for the club subset  $D = \Lambda$  (and  $K$  is therefore uniformly Eberlein compact by Lemma 3.4(ii)), but  $K$  also satisfies (II') because  $\Lambda \subseteq S$ , and each club subset  $E$  of  $[0, \omega_1)$  intersects  $\Lambda$ , so that  $S \cap E \neq \emptyset$ . Hence conditions (I) and (II') are not mutually exclusive.

**Proposition 3.10.** *Every uniformly Eberlein compact space which contains a dense subset of cardinality at most  $\aleph_1$  is homeomorphic to a weakly\* compact subset of  $C_0[0, \omega_1]^*$ .*

*Proof.* As above, let  $\Lambda$  be the set of all countable limit ordinals. Then every uniformly Eberlein compact space containing a dense subset of cardinality at most  $\aleph_1$  embeds in the closed unit ball  $B_{\ell_2(\Lambda)}$  of the Hilbert space  $\ell_2(\Lambda) = \{f : \Lambda \rightarrow \mathbb{K} : \sum_{\alpha \in \Lambda} |f(\alpha)|^2 < \infty\}$ , equipped with the weak topology. Hence it will suffice to prove that the mapping given by

$$\theta : f \mapsto \sum_{\alpha \in \Lambda} f(\alpha) |f(\alpha)| (\delta_\alpha - \delta_{\alpha+1}), \quad B_{\ell_2(\Lambda)} \rightarrow C_0[0, \omega_1]^*, \quad (3.15)$$

is a weakly-weakly\* continuous injection. The injectivity is clear. Suppose that the net  $(f_j)_{j \in J}$  in  $B_{\ell_2(\Lambda)}$  converges weakly to  $f$ , and let  $\varepsilon > 0$  and  $g \in C_0[0, \omega_1]$  be given. Since the indicator functions  $\mathbf{1}_{[0, \alpha]}$  for  $\alpha \in [0, \omega_1)$  span a norm-dense subspace of  $C_0[0, \omega_1]$ , it suffices to consider the case where  $g = \mathbf{1}_{[0, \alpha]}$  for some  $\alpha \in [0, \omega_1)$ . Now

$$\langle \mathbf{1}_{[0, \alpha]}, \theta(h) \rangle = \begin{cases} h(\alpha) |h(\alpha)| & \text{if } \alpha \in \Lambda \\ 0 & \text{otherwise} \end{cases} \quad (h \in \ell_2(\Lambda)),$$

so that we may suppose that  $\alpha \in \Lambda$ . Choosing  $j_0 \in J$  such that  $|f(\alpha) - f_j(\alpha)| < \varepsilon/2$  whenever  $j \geq j_0$ , we obtain

$$\begin{aligned} |\langle \mathbf{1}_{[0,\alpha]}, \theta(f) - \theta(f_j) \rangle| &= |f(\alpha)|f(\alpha)| - f_j(\alpha)|f_j(\alpha)| \\ &\leq 2 \max\{|f(\alpha)|, |f_j(\alpha)|\} |f(\alpha) - f_j(\alpha)| < \varepsilon \quad (j \geq j_0), \end{aligned}$$

which proves that  $(\theta(f_j))_{j \in J}$  converges weakly\* to  $\theta(f)$ .  $\square$

**Remark 3.11.** The mapping  $\theta: \ell_2(\Lambda) \rightarrow C_0[0, \omega_1]^*$  given by (3.15) is clearly not linear. In fact, no weakly-weakly\* continuous, linear mapping  $T: \ell_2(\Lambda) \rightarrow C_0[0, \omega_1]^*$  is injective. To verify this, we first observe that  $T$  is weakly compact because its domain is reflexive, and hence compact because its codomain has the Schur property. Moreover, using the reflexivity of  $\ell_2(\Lambda)$  once more, we see that the weak-weak\* continuity of  $T$  implies that  $T = S^*$  for some operator  $S: C_0[0, \omega_1] \rightarrow \ell_2(\Lambda)$ . Schauder's theorem then shows that  $S$  is compact, so that it has separable range. In particular, the range of  $S$  is not dense in  $\ell_2(\Lambda)$ , and therefore  $T = S^*$  is not injective.

#### 4. OPERATOR THEORY ON $C_0[0, \omega_1]$

The following lemma represents the core of our proof of Theorem 1.2.

**Lemma 4.1.** *Let  $X$  be a Banach space, and suppose that there exists a surjective operator  $T: C_0[0, \omega_1] \rightarrow X$ . Then exactly one of the following two alternatives holds:*

- (I) *either  $X$  embeds in a Hilbert-generated Banach space; or*
- (II) *there exists a club subset  $D$  of  $[0, \omega_1]$  such that the restriction of  $T$  to the subspace  $\mathcal{R}_D$  given by (2.7) is bounded below, and  $T[\mathcal{R}_D]$  is complemented in  $X$ .*

*Proof.* Let  $B_{X^*}$  be the closed unit ball of  $X^*$ . The weak\* continuity of  $T^*$  implies that the subset  $K = T^*[B_{X^*}]$  of  $C_0[0, \omega_1]^*$  is weakly\* compact, so by Theorem 1.1, we have:

- (I) *either  $K$  is uniformly Eberlein compact; or*
- (II) *there exist  $\rho \in K$ ,  $\lambda \in \mathbb{K} \setminus \{0\}$ , and a club subset  $D$  of  $[0, \omega_1]$  such that  $\rho + \lambda\delta_\alpha \in K$  for each  $\alpha \in D$ .*

Since  $T^*$  is injective by the assumption, its restriction to  $B_{X^*}$  is a weak\* homeomorphism onto  $K$ . Hence, in the first case,  $B_{X^*}$  is also uniformly Eberlein compact, and so  $X$  embeds in a Hilbert-generated Banach space by Theorem 2.3.

Otherwise we can choose functionals  $g, g_\alpha \in B_{X^*}$  such that  $T^*g = \rho$  and  $T^*g_\alpha = \lambda\delta_\alpha + \rho$  for each  $\alpha \in D$ . Given  $x \in X$ , we define a mapping  $Sx: [0, \omega_1] \rightarrow \mathbb{K}$  by

$$(Sx)(\alpha) = \langle x, g_{\pi_D(\alpha)} - g \rangle \quad (\alpha \in [0, \omega_1]), \quad (4.1)$$

where  $\pi_D: [0, \omega_1] \rightarrow D$  is the retraction defined by (2.5). Suppose that  $x = Tf$  for some  $f \in C_0[0, \omega_1]$ . Then, for each  $\alpha \in [0, \omega_1]$ , we have

$$(Sx)(\alpha) = \langle Tf, g_{\pi_D(\alpha)} - g \rangle = \langle f, T^*g_{\pi_D(\alpha)} - T^*g \rangle = \lambda f(\pi_D(\alpha)) = \lambda(P_D f)(\alpha), \quad (4.2)$$

so that  $Sx = \lambda P_D f$ , where  $P_D$  is the projection defined in Lemma 2.12(iv). Since  $\lambda P_D f \in C_0[0, \omega_1]$ , we see that (4.1) defines a mapping  $S: X \rightarrow C_0[0, \omega_1]$ , which is linear by the linearity of the functionals  $g_{\pi_D(\alpha)}$  and  $g$ . Moreover,  $S$  is bounded because the Open

Mapping Theorem implies that there exists a constant  $C > 0$ , dependent only on the surjective operator  $T$ , such that, for each  $x \in X$ , there exists  $f \in C_0[0, \omega_1)$  with  $\|f\| \leq C\|x\|$  and  $Tf = x$ . Then we have  $Sx = \lambda P_D f$  by (4.2), and hence  $\|Sx\| = |\lambda| \|P_D f\| \leq |\lambda| C\|x\|$ , as desired.

Another application of (4.2) shows that  $STP_D = \lambda P_D$  because

$$(STP_D)f = S(T(P_D f)) = \lambda P_D(P_D f) = \lambda P_D f \quad (f \in C_0[0, \omega_1)).$$

It is now straightforward to verify that  $T|_{\mathcal{R}_D}$  is bounded below by  $|\lambda|/\|S\|$  and that the operator  $\lambda^{-1}TP_DS \in \mathcal{B}(X)$  is a projection with range  $T[\mathcal{R}_D]$ .

Finally, the two conditions are mutually exclusive because (II) implies that  $T$  induces an isomorphic embedding of  $\mathcal{R}_D \cong C_0[0, \omega_1)$  in  $X$ , and hence (I) cannot be satisfied by Theorem 2.10.  $\square$

*Proof of Theorem 1.2.* Suppose that (I) is not satisfied. Then, by Lemma 4.1, we can find a club subset  $D$  of  $[0, \omega_1)$  and an idempotent operator  $Q$  on  $X$  such that  $T|_{\mathcal{R}_D}$  is bounded below and  $Q[X] = T[\mathcal{R}_D]$ . Hence the identity operator on  $C_0[0, \omega_1)$  factors through  $T$  by Lemmas 2.1 and 2.12(iv).

We shall complete the proof that (II) is satisfied by showing that  $\ker Q$  embeds in a Hilbert-generated Banach space. Assume the contrary, and apply Lemma 4.1 to the surjective operator  $U: f \mapsto (I_X - Q)Tf, C_0[0, \omega_1) \rightarrow \ker Q$ , to obtain a club subset  $E$  of  $[0, \omega_1)$  such that  $U|_{\mathcal{R}_E}$  is bounded below by  $\varepsilon > 0$ , say. Then we have

$$\varepsilon\|f\| \leq \|Uf\| = \|(I_X - Q)Tf\| = 0 \quad (f \in \mathcal{R}_D \cap \mathcal{R}_E),$$

so that  $\mathcal{R}_D \cap \mathcal{R}_E = \{0\}$ . This, however, contradicts Lemma 2.15.

Theorem 2.10 shows that conditions (I) and (II) are mutually exclusive.  $\square$

**Remark 4.2.** Not all quotients of  $C_0[0, \omega_1)$  are subspaces of  $C_0[0, \omega_1)$ . This follows from a result of Alspach [1], which says that  $C[0, \omega^\omega]$  has a quotient  $X$  which does not embed in  $C[0, \alpha]$  for any countable ordinal  $\alpha$ . Since  $C_0[0, \omega_1)$  contains a complemented copy of  $C[0, \omega^\omega]$ ,  $X$  is also a quotient of  $C_0[0, \omega_1)$ . However,  $X$  does not embed in  $C_0[0, \omega_1)$  because  $X$  is separable (being a quotient of a separable space), and each separable subspace of  $C_0[0, \omega_1)$  embeds in  $C[0, \alpha]$  for some countable ordinal  $\alpha$  (e.g., see [18, Lemma 4.2]).

**Lemma 4.3.** *Exactly one of the following two alternatives holds for each continuous mapping  $\theta$  from  $[0, \omega_1]$  into a Hausdorff space  $K$ :*

- (I) *either  $[0, \omega_1]$  contains a closed, uncountable subset  $D$  such that  $\theta|_D$  is constant; or*
- (II)  *$[0, \omega_1]$  contains a closed, uncountable subset  $E$  such that  $\theta|_E$  is injective, and hence a homeomorphism onto  $\theta[E]$ .*

*Proof.* Suppose that  $K$  contains a point  $x$  whose pre-image under  $\theta$  is uncountable. Then (I) is satisfied for  $D = \theta^{-1}[\{x\}]$ .

Otherwise each point of  $K$  has countable pre-image under  $\theta$ . In this case we shall inductively construct a strictly increasing transfinite sequence  $(\alpha_\xi)_{\xi < \omega_1}$  in  $[1, \omega_1)$  such that

$$\{\alpha \in [0, \omega_1) : \theta(\alpha) = \theta(\omega_1) \text{ or } \theta(\alpha) = \theta(\alpha_\eta) \text{ for some } \eta < \xi\} \subseteq [0, \alpha_\xi] \quad (\xi \in [0, \omega_1)) \quad (4.3)$$

and

$$\sup_{\eta < \xi} \alpha_\eta \in \{\alpha_\eta : \eta \leq \xi\} \quad (\xi \in [1, \omega_1)). \quad (4.4)$$

To start the induction, let  $\alpha_0 = \sup\{\alpha \in [0, \omega_1) : \theta(\alpha) = \theta(\omega_1)\} + 1 \in [1, \omega_1)$ .

Now assume inductively that, for some  $\xi \in [1, \omega_1)$ , a strictly increasing sequence  $(\alpha_\eta)_{\eta < \xi}$  of countable ordinals has been chosen in accordance with (4.3) and (4.4), and define

$$\alpha_\xi = \sup\{\alpha + 1 : \theta(\alpha) = \theta(\alpha_\eta) \text{ for some } \eta < \xi\} \in \left[\sup_{\eta < \xi} (\alpha_\eta + 1), \omega_1\right).$$

Then (4.3) is certainly satisfied for  $\xi$ . To verify (4.4), let  $\beta = \sup_{\eta < \xi} \alpha_\eta$ . The conclusion is clear if this supremum is attained. Otherwise  $\xi$  is a limit ordinal, and we claim that  $\beta = \alpha_\xi$ . Since  $\beta \leq \alpha_\xi$ , it suffices to show that  $\alpha + 1 \leq \beta$  whenever  $\alpha \in [0, \omega_1)$  satisfies  $\theta(\alpha) = \theta(\alpha_\eta)$  for some  $\eta < \xi$ . Now  $\eta + 1 < \xi$  because  $\xi$  is a limit ordinal, so that (4.3) holds for  $\eta + 1$  by the induction hypothesis. Hence  $\alpha + 1 \leq \alpha_{\eta+1} < \beta$ , and the induction continues.

Let  $E = \{\alpha_\xi : \xi < \omega_1\} \cup \{\omega_1\}$ , which is uncountable because  $(\alpha_\xi)_{\xi < \omega_1}$  is strictly increasing, and closed by (4.4). Moreover,  $\theta|_E$  is injective by (4.3), so that (II) is satisfied.

Finally, to see that conditions (I) and (II) are mutually exclusive, assume towards a contradiction that  $[0, \omega_1]$  contains closed, uncountable subsets  $D$  and  $E$  such that  $\theta|_D$  is constant and  $\theta|_E$  is injective. Then  $D \cap E$  is uncountable and contains  $\omega_1$ , so that  $\theta(\alpha) = \theta(\omega_1)$  for each  $\alpha \in D \cap E$  by the choice of  $D$ . This, however, contradicts the injectivity of  $\theta|_E$ .  $\square$

*Proof of Theorem 1.3.* Let  $T \in \mathcal{B}(C_0[0, \omega_1])$ . The following “function-free” reformulation of (1.1) will enable us to simplify certain calculations somewhat:

$$T^* \delta_\alpha = \varphi(T) \delta_\alpha \quad (\alpha \in D). \quad (4.5)$$

To prove that there exist  $\varphi(T) \in \mathbb{K}$  and a club subset  $D$  of  $[0, \omega_1)$  such that this identity is satisfied, we consider the composite mapping  $\theta$  given by

$$\begin{array}{ccc} [0, \omega_1] & \xrightarrow{\quad \theta \quad} & C_0[0, \omega_1]^* \\ & \searrow \sigma_{1,0} \quad \nearrow T^* & \\ & C_0[0, \omega_1]^*, & \end{array}$$

where both copies of  $C_0[0, \omega_1]^*$  are equipped with the weak\* topology, and  $\sigma_{1,0}$  is the injection defined in Lemma 3.1, that is,  $\sigma_{1,0}(\alpha) = \delta_\alpha$  for  $\alpha \in [0, \omega_1)$  and  $\sigma_{1,0}(\omega_1) = 0$ . Since  $\sigma_{1,0}$  and  $T^*$  are continuous, so is  $\theta$ . We may therefore apply Lemma 4.3 to conclude that  $[0, \omega_1]$  contains a closed, uncountable subset  $E$  such that either  $\theta|_E$  is constant, or  $\theta|_E$  is injective. Note that  $\omega_1 \in E$ , and  $E \setminus \{\omega_1\}$  is a club subset of  $[0, \omega_1)$ .

If  $\theta|_E$  is constant, then we have

$$T^* \delta_\alpha = (T^* \circ \sigma_{1,0})(\alpha) = \theta(\alpha) = \theta(\omega_1) = (T^* \circ \sigma_{1,0})(\omega_1) = T^* 0 = 0 \quad (\alpha \in E \setminus \{\omega_1\}),$$

so that (4.5) is satisfied for  $D = E \setminus \{\omega_1\}$  and  $\varphi(T) = 0$ .

Otherwise  $\theta|_E$  is injective, in which case  $\tilde{\theta}: \alpha \mapsto \theta(\alpha)$ ,  $E \rightarrow \theta[E]$ , is a homeomorphism. Since  $E$  is homeomorphic to  $[0, \omega_1]$ , which is not Eberlein compact, Theorem 1.1 implies that there exist  $\rho \in \theta[E]$ ,  $\varphi(T) \in \mathbb{K} \setminus \{0\}$  and a club subset  $F$  of  $[0, \omega_1)$  such that  $\rho + \varphi(T)\delta_\alpha \in \theta[E]$  for each  $\alpha \in F$ ; let  $\eta_\alpha = \tilde{\theta}^{-1}(\rho + \varphi(T)\delta_\alpha)$ . Then  $(\eta_\alpha)_{\alpha \in F}$  is an uncountable net of distinct elements of  $E$ , and it converges to  $\tilde{\theta}^{-1}(\rho)$ . The only possible limit of such a net is  $\omega_1$ , so that  $\rho = \theta(\omega_1) = 0$ , and we have

$$T^*\delta_{\eta_\alpha} = \theta(\eta_\alpha) = \varphi(T)\delta_\alpha \quad (\alpha \in F). \quad (4.6)$$

We shall now show that  $D = \{\alpha \in [0, \omega_1) : T^*\delta_\alpha = \varphi(T)\delta_\alpha\}$  is a club subset of  $[0, \omega_1)$ ; this will complete the existence part of the proof because (4.5) is evidently satisfied for this choice of  $D$ .

Suppose that  $(\alpha_j)_{j \in J}$  is a net in  $D$  converging to  $\alpha \in [0, \omega_1)$ . The net  $(\delta_{\alpha_j})_{j \in J}$  is then weakly\* convergent with limit  $\delta_\alpha$ . Hence, by the weak\* continuity of  $T^*$ , we have

$$T^*\delta_\alpha = \text{w}^*\text{-}\lim_j T^*\delta_{\alpha_j} = \text{w}^*\text{-}\lim_j \varphi(T)\delta_{\alpha_j} = \varphi(T)\delta_\alpha,$$

so that  $\alpha \in D$ , which proves that  $D$  is closed.

To prove that  $D$  is unbounded, let  $\gamma \in [0, \omega_1)$ . For each  $\beta \in [0, \omega_1)$ , the sets  $F \cap [0, \beta]$  and  $\{\alpha \in F : \eta_\alpha \leq \beta\}$  are both countable, so that the complement of their union, which is equal to  $\{\alpha \in F \cap [\beta + 1, \omega_1) : \eta_\alpha > \beta\}$ , is uncountable, and thus non-empty. Using this, we can inductively construct a sequence  $(\alpha_n)_{n \in \mathbb{N}}$  in  $F \cap [\gamma + 1, \omega_1)$  such that

$$\max\{\alpha_n, \eta_{\alpha_n}\} < \min\{\alpha_{n+1}, \eta_{\alpha_{n+1}}\} \quad (n \in \mathbb{N}).$$

Let  $\alpha = \sup_{n \in \mathbb{N}} \alpha_n \in F \cap [\gamma + 1, \omega_1)$ . Then both of the sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\eta_{\alpha_n})_{n \in \mathbb{N}}$  converge to  $\alpha$ , so that  $(\delta_{\alpha_n})_{n \in \mathbb{N}}$  and  $(\delta_{\eta_{\alpha_n}})_{n \in \mathbb{N}}$  both weakly\* converge to  $\delta_\alpha$ , and consequently

$$T^*\delta_\alpha = \text{w}^*\text{-}\lim_n T^*\delta_{\eta_{\alpha_n}} = \text{w}^*\text{-}\lim_n \varphi(T)\delta_{\alpha_n} = \varphi(T)\delta_\alpha$$

by the weak\* continuity of  $T^*$  and (4.6), so that  $\alpha \in D$ , which is therefore unbounded.

We shall next prove that the scalar  $\varphi(T)$  is uniquely determined by the operator  $T$ . Suppose that  $\varphi_1(T)$  and  $\varphi_2(T)$  are scalars such that

$$(Tf)(\alpha) = \varphi_j(T)f(\alpha) \quad (\alpha \in D_j, f \in C_0[0, \omega_1), j = 1, 2)$$

for some club subsets  $D_1$  and  $D_2$  of  $[0, \omega_1)$ . Then  $D_1 \cap D_2$  is a club subset, and hence non-empty. Taking  $\alpha \in D_1 \cap D_2$ , we obtain

$$(T\mathbf{1}_{[0, \alpha]})(\alpha) = \varphi_j(T)\mathbf{1}_{[0, \alpha]}(\alpha) = \varphi_j(T) \quad (j = 1, 2),$$

so that  $\varphi_1(T) = \varphi_2(T)$ , as required.

Consequently, we can define a mapping  $\varphi: T \mapsto \varphi(T)$ ,  $\mathcal{B}(C_0[0, \omega_1)) \rightarrow \mathbb{K}$ , which is non-zero because  $\varphi(I_{C_0[0, \omega_1)}) = 1$ . To see that  $\varphi$  is an algebra homomorphism, let  $\lambda \in \mathbb{K}$  and  $T_1, T_2 \in \mathcal{B}(C[0, \omega_1))$  be given, and take club subsets  $D_1$  and  $D_2$  of  $[0, \omega_1)$  such that

$$T_j^*\delta_\alpha = \varphi(T_j)\delta_\alpha \quad (\alpha \in D_j, j = 1, 2).$$

Then, for each  $\alpha$  belonging to the club subset  $D_1 \cap D_2$  of  $[0, \omega_1)$ , we have

$$(\lambda T_1 + T_2)^*\delta_\alpha = \lambda T_1^*\delta_\alpha + T_2^*\delta_\alpha = (\lambda \varphi(T_1) + \varphi(T_2))\delta_\alpha$$

and

$$(T_1 T_2)^* \delta_\alpha = T_2^*(T_1^* \delta_\alpha) = T_2^*(\varphi(T_1) \delta_\alpha) = \varphi(T_1) \varphi(T_2) \delta_\alpha,$$

so that  $\varphi(\lambda T_1 + T_2) = \lambda \varphi(T_1) + \varphi(T_2)$  and  $\varphi(T_1 T_2) = \varphi(T_1) \varphi(T_2)$  by the definition of  $\varphi$ .  $\square$

**Lemma 4.4.** *Let  $T$  be an operator on  $C_0[0, \omega_1)$  such that*

$$(Tf)(\alpha) = 0 \quad (f \in C_0[0, \omega_1), \alpha \in D) \quad (4.7)$$

*for some club subset  $D$  of  $[0, \omega_1)$ . Then the range of  $T$  is contained in the kernel of the projection  $P_D$  introduced in Lemma 2.12(iv).*

*Proof.* This is immediate from (2.6) and (4.7).  $\square$

*Proof of Theorem 1.4.* We begin by showing that conditions (b) and (c) are equivalent. This relies on the fact that, by (2.2), we have

$$T_{\alpha, \beta} = T^* \delta_\alpha(\{\beta\}) \quad (\alpha, \beta \in [0, \omega_1)). \quad (4.8)$$

To see that (b) implies (c), suppose that  $D$  is a club subset of  $[0, \omega_1)$  such that  $T_{\alpha, \alpha} = 0$  for each  $\alpha \in D$ , and take a club subset  $E$  of  $[0, \omega_1)$  such that  $T^* \delta_\alpha = \varphi(T) \delta_\alpha$  for each  $\alpha \in E$ . Then, choosing  $\alpha \in D \cap E$  (which is possible because  $D \cap E$  is a club subset, and hence non-empty), we obtain

$$0 = T_{\alpha, \alpha} = T^* \delta_\alpha(\{\alpha\}) = \varphi(T) \delta_\alpha(\{\alpha\}) = \varphi(T),$$

which shows that (c) is satisfied.

Conversely, suppose that  $\varphi(T) = 0$ , so that there exists a club subset  $D$  of  $[0, \omega_1)$  such that  $T^* \delta_\alpha = 0$  for each  $\alpha \in D$ . Then (4.8) immediately shows that  $T_{\alpha, \alpha} = 0$  for each  $\alpha \in D$ , so that (b) is satisfied.

We shall next prove that conditions (c)–(h) are equivalent.

The implication (c) $\Rightarrow$ (d) follows immediately from Lemma 4.4 and Corollary 2.13.

(d) $\Rightarrow$ (e). Suppose that we have  $T = SR$  for some operators  $R: C_0[0, \omega_1) \rightarrow C_0(L_0)$  and  $S: C_0(L_0) \rightarrow C_0[0, \omega_1)$ . Then  $R \in \mathcal{H}\mathcal{G}(C_0[0, \omega_1), C_0(L_0))$  by Corollary 2.9, and hence  $T \in \mathcal{H}\mathcal{G}(C_0[0, \omega_1))$  by Lemma 2.5(i).

The implication (e) $\Rightarrow$ (f) is clear because each Hilbert-generated Banach space is weakly compactly generated.

The implications (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (c) are all proved by contraposition.

(f) $\Rightarrow$ (g). Suppose that  $TU$  is an isomorphism onto its range for some operator  $U$  on  $C_0[0, \omega_1)$ . Then, by Theorem 2.10, the range of  $TU$ , and hence the range of  $T$ , cannot be contained in any weakly compactly generated Banach space.

(g) $\Rightarrow$ (h). Suppose that  $STR = I_{C_0[0, \omega_1)}$  for some operators  $R$  and  $S$  on  $C_0[0, \omega_1)$ . The operator  $TR$  is then bounded below, and hence an isomorphism onto its range, so that  $T$  fixes a copy of  $C_0[0, \omega_1)$ .

(h) $\Rightarrow$ (c). Suppose that  $\varphi(T) \neq 0$ . By rescaling, we may suppose that  $\varphi(T) = 1$ , so that (1.1) is satisfied for some club subset  $D$  of  $[0, \omega_1)$ . Then, in the notation of Lemma 2.12,



we have a commutative diagram, which implies that (h) is not satisfied:

$$\begin{array}{ccc}
 C_0[0, \omega_1) & \xrightarrow{I_{C_0[0, \omega_1)}} & C_0[0, \omega_1) \\
 \downarrow \text{dashed} & \searrow U_D^{-1} & \nearrow U_D \\
 & C_0(D) \xrightarrow{I_{C_0(D)}} C_0(D) & \\
 \nearrow S_D & & \nwarrow R_D \\
 C_0[0, \omega_1) & \xrightarrow{T} & C_0[0, \omega_1)
 \end{array}$$

Indeed, the commutativity of the upper trapezium is clear, while for the lower one, we find

$$(R_D T S_D g)(\alpha) = T(g \circ \pi_D)(\alpha) = (g \circ \pi_D)(\alpha) = g(\alpha) \quad (g \in C_0(D), \alpha \in D).$$

Finally, to see that conditions (a) and (c) are equivalent, we note that, on the one hand, the Loy–Willis ideal  $\mathcal{M}$  is a maximal ideal of  $\mathcal{B}(C_0[0, \omega_1))$  by its definition. On the other, the implication (h) $\Rightarrow$ (c), which has just been established, shows that the identity operator belongs to the ideal generated by any operator not in  $\ker \varphi$ , so that  $\ker \varphi$  is the *unique* maximal ideal of  $\mathcal{B}(C_0[0, \omega_1))$ . Hence  $\ker \varphi = \mathcal{M}$ .  $\square$

As we have already noted in the final paragraph of the proof above, Theorem 1.4 has the following important consequence, which generalizes [18, Theorem 1.1].

**Corollary 4.5.** *The ideal  $\mathcal{M} = \ker \varphi = \mathcal{HG}(C_0[0, \omega_1))$  is the unique maximal ideal of  $\mathcal{B}(C_0[0, \omega_1))$ .*

*Proof of Corollary 1.7.* Let  $\Gamma$  denote the set of all club subsets of  $[0, \omega_1)$ , ordered by reverse inclusion. This order is filtering upward because  $D \cap E \in \Gamma$  is a majorant for any pair  $D, E \in \Gamma$ , and hence Theorem 1.4, Lemma 2.12(iv) and Corollary 2.13 imply that

$$Q_D = I_{C_0[0, \omega_1)} - P_D \in \mathcal{M} \quad (D \in \Gamma)$$

defines a net of projections, each having norm at most two. For each  $T \in \mathcal{M}$ , we have  $P_D T = 0$  for some club subset  $D$  of  $[0, \omega_1)$  by Lemma 4.4. Equation (2.6) then shows that  $P_E T = 0$  for each  $E \subseteq D$ ; that is,  $Q_E T = T$  whenever  $E \supseteq D$ .  $\square$

**Example 4.6.** Consider the Hilbert space  $H = \{f: [0, \omega_1) \rightarrow \mathbb{K} : \sum_{\alpha < \omega_1} |f(\alpha)|^2 < \infty\}$ . The work of Gramsch [13] and Luft [23] shows that the set  $\mathcal{X}(H)$  of operators on  $H$  having separable range is the unique maximal ideal of  $\mathcal{B}(H)$ . (In fact, Gramsch and Luft proved that the entire lattice of closed ideals of  $\mathcal{B}(H)$  is given by

$$\{0\} \subset \mathcal{K}(H) \subset \mathcal{X}(H) \subset \mathcal{B}(H),$$

but we do not require the full strength of their result.) Since  $\mathcal{B}(H)$  is a  $C^*$ -algebra, each of its closed ideals has a bounded two-sided approximate identity consisting of positive contractions. The purpose of this example is to show that, in the case of  $\mathcal{X}(H)$ , we have a bounded two-sided approximate identity  $(P_L)_{L \in \Gamma}$  consisting of contractive, self-adjoint

projections such that  $P_L T = T = T P_L$  eventually for each  $T \in \mathcal{X}(H)$ . We note in passing that algebras which contain a net with this property have been studied in a purely algebraic context by Ara and Perera [4, Definition 1.4] and Pedersen and Perera [25, Section 4].

Let  $\Gamma$  denote the set of all closed, separable subspaces of  $H$ , ordered by inclusion. This order is filtering upward because  $\overline{L + M} \in \Gamma$  majorizes the pair  $L, M \in \Gamma$ . For  $L \in \Gamma$ , let  $P_L \in \mathcal{X}(H)$  be the orthogonal projection which has range  $L$ . Suppose that  $T \in \mathcal{X}(H)$ , and denote by  $T^*$  the Hilbert-space adjoint of  $T$ . We have  $T^* \in \mathcal{X}(H)$  because each closed ideal of a  $C^*$ -algebra is self-adjoint, and therefore  $M = \overline{T[H]} + \overline{T^*[H]}$  belongs to  $\Gamma$ . Now, for each  $L \in \Gamma$  such that  $L \supseteq M$ , we see that  $P_L T = T$  and  $P_L T^* = T^*$ , from which the desired conclusion follows by taking the adjoint of the latter equation.

*Proof of Corollary 1.8.* Assume towards a contradiction that, for some natural numbers  $m > n$ , there exists either an operator  $R: C_0[0, \omega_1]^m \rightarrow C_0[0, \omega_1]^n$  which is bounded below, or an operator  $T: C_0[0, \omega_1]^n \rightarrow C_0[0, \omega_1]^m$  which is surjective. We shall focus on the first case; the other is very similar. The proof is best explained if we represent the operator  $R: C_0[0, \omega_1]^m \rightarrow C_0[0, \omega_1]^n$  by the operator-valued  $(n \times m)$ -matrix  $(R_{j,k})_{j,k=1}^{n,m}$  given by  $R_{j,k} = Q_j^{(n)} R J_k^{(m)} \in \mathcal{B}(C_0[0, \omega_1])$ , where  $Q_j^{(n)}: C_0[0, \omega_1]^n \rightarrow C_0[0, \omega_1]$  and  $J_k^{(m)}: C_0[0, \omega_1] \rightarrow C_0[0, \omega_1]^m$  denote the  $j^{\text{th}}$  coordinate projection and  $k^{\text{th}}$  coordinate embedding, respectively.

Using elementary column operations, we can reduce the scalar-valued matrix  $S = (\varphi(R_{j,k}))_{j,k=1}^{n,m}$  to column-echelon form; that is, we can find an invertible, scalar-valued  $(m \times m)$ -matrix  $U$  such that  $SU$  has column-echelon form. Since  $m > n$ , the final column of  $SU$  must be zero. Consequently, each operator in the final column of the matrix  $RU$  belongs to  $\mathcal{M} = \mathcal{H}\mathcal{G}(C_0[0, \omega_1])$ , so that  $RU J_m^{(m)} \in \mathcal{H}\mathcal{G}(C_0[0, \omega_1], C_0[0, \omega_1]^n)$ . This, however, contradicts Theorem 2.10 because each of the operators  $J_m^{(m)}$ ,  $U$  and  $R$  is bounded below, and therefore the range of  $RU J_m^{(m)}$  is isomorphic to its domain  $C_0[0, \omega_1]$ .  $\square$

*Proof of Corollary 1.9.* Since  $P$  is idempotent, we have  $\varphi(P) \in \{0, 1\}$ . We shall consider the case where  $\varphi(P) = 0$ ; the case where  $\varphi(P) = 1$  is similar, just with  $P$  and  $I_{C_0[0, \omega_1]} - P$  interchanged. Let  $X = \ker P$  and  $Y = P[C_0[0, \omega_1]]$ . Lemma 4.4 implies that  $Y$  is contained in  $\ker P_D$  for some club subset  $D$  of  $[0, \omega_1]$ . By Corollary 2.13,  $\ker P_D$  is isomorphic to a complemented subspace of  $C_0(L_0)$ , so that the same is true for  $Y$ , say  $C_0(L_0) \cong Y \oplus Z$  for some Banach space  $Z$ . Writing  $c_0(\mathbb{N}, W)$  for the  $c_0$ -direct sum of countably many copies of a Banach space  $W$  and using Corollary 2.7, we obtain

$$C_0(L_0) \cong c_0(\mathbb{N}, C_0(L_0)) \cong c_0(\mathbb{N}, Y \oplus Z) \cong Y \oplus c_0(\mathbb{N}, Y \oplus Z) \cong Y \oplus C_0(L_0).$$

Consequently  $C_0[0, \omega_1] \cong C_0[0, \omega_1] \oplus Y$  because  $C_0[0, \omega_1]$  contains a complemented subspace isomorphic to  $C_0(L_0)$  by Corollary 2.14. Theorem 1.2 implies that  $X$  contains a complemented subspace which is isomorphic to  $C_0[0, \omega_1]$ , so that  $X \cong W \oplus C_0[0, \omega_1]$  for some Banach space  $W$ , and hence we have

$$X \cong W \oplus C_0[0, \omega_1] \cong W \oplus C_0[0, \omega_1] \oplus Y \cong X \oplus Y = C_0[0, \omega_1],$$

as required.  $\square$

*Proof of Corollary 1.10.* Let  $U$  be an isomorphism of  $C_0[0, \omega_1)$  onto the closed subspace  $X$ , and consider the operator  $T = JU \in \mathcal{B}(C_0[0, \omega_1))$ , where  $J: X \rightarrow C_0[0, \omega_1)$  denotes the natural inclusion. Then  $T$  fixes a copy of  $C_0[0, \omega_1)$ . Hence, by Theorem 1.4, we can find operators  $R$  and  $S$  on  $C_0[0, \omega_1)$  such that  $STR = I_{C_0[0, \omega_1)}$ . This implies that  $TR$  is an isomorphism of  $C_0[0, \omega_1)$  onto its range  $Y$ , which is contained in  $X$ , and  $TRS$  is a projection of  $C_0[0, \omega_1)$  onto  $Y$ .  $\square$

*Proof of Corollary 1.11.* This proof follows closely that of [31, Proposition 8], where any unexplained terminology can also be found.

We begin by showing that Willis's ideal of compressible operators on  $C_0[0, \omega_1)$ , as defined in [31, p. 252], is equal to the Loy–Willis ideal  $\mathcal{M}$ . Indeed, [31, Proposition 2] and Corollary 1.8 show that the identity operator on  $C_0[0, \omega_1)$  is not compressible, so that the ideal of compressible operators is proper, and hence contained in  $\mathcal{M}$ . Conversely, each operator  $T \in \mathcal{M}$  is compressible by [31, Proposition 1] because  $T$  factors through a complemented subspace  $X$  of  $C_0[0, \omega_1)$  such that  $X \cong C_0(L_0)$ , and Corollary 2.7 implies that  $X$  is isomorphic to the  $c_0$ -direct sum of countably many copies of itself.

Next, we observe that null sequences in  $\mathcal{M}$  factor, in the sense that for each norm-null sequence  $(T_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}$ , there are  $T \in \mathcal{M}$  and a norm-null sequence  $(S_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}$  such that  $T_n = TS_n$  for each  $n \in \mathbb{N}$ . This is a standard consequence of Cohen's Factorization Theorem (e.g., see [8, Corollary I.11.2]), which applies because  $\mathcal{M}$  has a bounded left approximate identity. However, we do not need Cohen's Factorization Theorem to verify this. Indeed, for each  $n \in \mathbb{N}$ , take a club subset  $D_n$  of  $[0, \omega_1)$  such that  $T_n^* \delta_\alpha = 0$  for each  $\alpha \in D_n$ . Then  $D = \bigcap_{n \in \mathbb{N}} D_n$  is a club subset of  $[0, \omega_1)$  such that  $T_n^* \delta_\alpha = 0$  for each  $\alpha \in D$  and  $n \in \mathbb{N}$ , and hence Corollary 2.13 and Lemma 4.4 imply that  $T = I_{C_0[0, \omega_1)} - P_D \in \mathcal{M}$  is an operator such that  $T_n = TT_n$  for each  $n \in \mathbb{N}$ .

Now consider an algebra homomorphism  $\theta$  from  $\mathcal{B}(C_0[0, \omega_1))$  into some Banach algebra  $\mathcal{C}$ . Then [31, Proposition 7] implies that the continuity ideal of  $\theta|_{\mathcal{M}}$  contains  $\mathcal{M}$ , so that the mapping  $S \mapsto \theta(TS)$ ,  $\mathcal{M} \rightarrow \mathcal{C}$ , is continuous for each fixed  $T \in \mathcal{M}$ . Since null sequences factor, as shown above, this proves the continuity of  $\theta|_{\mathcal{M}}$ , and thus of  $\theta$  because  $\mathcal{M}$  has finite codimension in  $\mathcal{B}(C_0[0, \omega_1))$ .  $\square$

*Proof of Corollary 1.13.* Each operator in  $\mathcal{M}$  factors through the Banach space  $C_0(L_0)$  by Theorem 1.4. Corollary 2.7 states that  $C_0(L_0)$  is isomorphic to the  $c_0$ -direct sum of countably many copies of itself, so that [21, Proposition 3.7] implies that each operator on  $C_0(L_0)$  is the sum of at most two commutators, and therefore each operator which factors through  $C_0(L_0)$  is the sum of at most three commutators by [21, Lemma 4.5].

Suppose that  $\tau$  is a trace on  $\mathcal{B}(C_0[0, \omega_1))$ . The first part of the proof implies that  $\mathcal{M} \subseteq \ker \tau$ , so that  $\tau(T + \lambda I_{C_0[0, \omega_1)}) = \lambda \tau(I_{C_0[0, \omega_1)}) = \varphi(T + \lambda I_{C_0[0, \omega_1)}) \tau(I_{C_0[0, \omega_1)})$  for each  $T \in \mathcal{M}$  and  $\lambda \in \mathbb{K}$ , as desired. The converse implication is clear.  $\square$

#### ACKNOWLEDGEMENTS

The authors would like to thank Richard Smith for a private exchange of emails [30], which was the seed that eventually grew into the Topological Dichotomy (Theorem 1.1).

Part of this work was carried out during a visit of the second author to Lancaster in February 2012, supported by a London Mathematical Society Scheme 2 grant (ref. 21101). The authors gratefully acknowledge this support. The second author was also partially supported by the Polish National Science Center research grant 2011/01/B/ST1/00657.

## REFERENCES

1. D. Alspach, *A quotient of  $C(\omega^\omega)$  which is not isomorphic to a subspace of  $C(\alpha)$ ,  $\alpha < \omega_1$* . Israel J. Math. 35 (1980), 49–60.
2. D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*. Israel J. Math. 27 (1977), 64–92.
3. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*. Ann. of Math. 88 (1968), 35–46.
4. P. Ara and F. Perera, *Multipliers of von Neumann regular rings*. Comm. Algebra 28 (2000), 3359–3385.
5. Y. Benyamini, M. E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*. Pacific J. Math. 70 (1977), 309–324.
6. Y. Benyamini and T. Starbird, *Embedding weakly compact sets into Hilbert space*. Israel J. Math. 23 (1976), 137–141.
7. C. Bessaga and A. Pełczyński, *Banach spaces non-isomorphic to their Cartesian squares. I*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 8 (1960), 77–80.
8. F. F. Bonsall and J. Duncan, *Complete normed algebras*. Ergebnisse der Mathematik und ihrer Grenzgebiete 80, Springer-Verlag, New York–Heidelberg, 1973.
9. J. B. Conway, *A Course in Functional Analysis*, 2<sup>nd</sup> edition. Graduate Texts in Mathematics 96, Springer-Verlag, 1990.
10. P. G. Dixon, *Approximate identities in normed algebras*. Proc. London Math. Soc. 26 (1973), 485–496.
11. M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 8, Springer-Verlag, New York, 2001.
12. G. Fodor, *Eine Bemerkung zur Theorie der regressiven Funktionen*. Acta Sci. Math. Szeged 17 (1956), 139–142.
13. B. Gramsch, *Eine Idealstruktur Banachscher Operatoralgebren*. J. Reine Angew. Math. 225 (1967), 97–115.
14. P. Hájek, V. Montesinos Santalucía, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 26, Springer-Verlag, New York, 2008.
15. T. Jech, *Set theory*. The third millennium edition, revised and expanded. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
16. T. Kania and T. Kochanek, *The ideal of weakly compactly generated operators acting on a Banach space*. J. Operator Th. (to appear).
17. T. Kania, P. Koszmider and N. J. Laustsen, *K-theory for the Banach algebra of bounded operators on the Banach space  $C[0, \omega_1]$* , in preparation.
18. T. Kania and N. J. Laustsen, *Uniqueness of the maximal ideal of the Banach algebra of bounded operators on  $C([0, \omega_1])$* . J. Funct. Anal. 262 (2012), 4831–4850.
19. K. Kunen, *Set theory*. An introduction to independence proofs. Studies in Logic and the Foundations of Mathematics 102, North-Holland, Amsterdam–New York, 1980.
20. N. J. Laustsen, *K-Theory for the Banach algebra of operators on James’s quasi-reflexive Banach spaces*. K-Theory 23 (2000), 115–127.
21. N. J. Laustsen, *Commutators of operators on Banach spaces*. J. Operator Th. 48 (2002), 503–514.

22. R. J. Loy and G. A. Willis, *Continuity of derivations on  $\mathcal{B}(E)$  for certain Banach spaces  $E$* . J. London Math. Soc. 40 (1989), 327–346.
23. E. Luft, *The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space*. Czechoslovak Math. J. 18 (1968), 595–605.
24. C. Ogden, *Homomorphisms from  $B(C(\omega_\eta))$* . J. London Math. Soc. 54 (1996), 346–358.
25. G. K. Pedersen and F. Perera, *Inverse limits of rings and multiplier rings*. Math. Proc. Cambridge Philos. Soc. 139 (2005), 207–228.
26. H. P. Rosenthal, *The Banach spaces  $C(K)$* . Handbook of the geometry of Banach spaces, Vol. 2, 1547–1602, North-Holland, Amsterdam, 2003.
27. W. Rudin, *Continuous functions on compact spaces without perfect subsets*. Proc. Amer. Math. Soc. 8 (1957), 39–42.
28. Z. Semadeni, *Banach spaces non-isomorphic to their Cartesian squares. II*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 8 (1960), 81–84.
29. N. A. Shanin, *A theorem from the general theory of sets*. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53 (1946), 399–400.
30. R. Smith, private communication.
31. G. Willis, *Compressible operators and the continuity of homomorphisms from algebras of operators*. Studia Math. 115 (1995), 251–259.

*E-mail address:* `t.kania@lancaster.ac.uk`

DEPARTMENT OF MATHEMATICS AND STATISTICS, FYLDE COLLEGE, LANCASTER UNIVERSITY, LANCASTER LA1 4YF, UNITED KINGDOM.

*E-mail address:* `P.Koszmider@Impan.pl`

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-956 WARSAWA, POLAND.

*E-mail address:* `n.laustsen@lancaster.ac.uk`

DEPARTMENT OF MATHEMATICS AND STATISTICS, FYLDE COLLEGE, LANCASTER UNIVERSITY, LANCASTER LA1 4YF, UNITED KINGDOM.